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Konstantin Avrachenkov — Urtzi Ayesta — Alexei Piunovskiy

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Konstantin Avrachenkov* , Urtzi Ayesta† , Alexei Piunovskiy‡

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Abstract: We study the interaction between the AIMD (Additive Increase Multiplicative Decrease) congestion control and a bottleneck router with Drop Tail buffer. We consider the problem in the framework of deterministic hybrid models. First, we show that the hybrid model of the interaction between the AIMD congestion control and bottleneck router always converges to a cyclic behavior. We characterize the cycles. Necessary and sufficient conditions for the absence of multiple jumps of congestion window in the same cycle are obtained. Then, we propose an analytical framework for the optimal choice of the router buffer size. We formulate the problem of the optimal router buffer size as a multi-criteria optimization problem, in which the Lagrange function corresponds to a linear combination of the average goodput and the average delay in the queue. The solution to the optimization problem provides further evidence that the buffer size should be reduced in the presence of traffic aggregation. Our analytical results are confirmed by simulations performed with Simulink and the NS simulator.

Key-words: TCP/IP modeling, router buffer sizing, deterministic hybrid model approach, multi-criteria optimization

* INRIA Sophia Antipolis, France, e-mail: k.avrachenkov@sophia.inria.fr .
† LAAS-CNRS, France, e-mail: urtzi@laas.fr .
‡ University of Liverpool, UK, e-mail: piunov@liverpool.ac.uk

Convergence et choix optimal de la taille du buffer du routeur pour le contrôle de congestion AIMD

Résumé : Nous étudions l'interaction entre le contrôle de congestion AIMD (Additive Increase Multiplicative Decrease) et le routeur avec le buffer de type Drop Tail. Nous considérons ce problème dans le cadre des modèles hybrides déterministes. D'abord, nous prouvons que le modèle hybride de l'interaction entre la côntrole de congestion AIMD et le routeur de goulot d'étranglement converge toujours à un comportement cyclique. Nous caractérisons les cycles. Des conditions nécessaires et suffisantes pour l'absence des sauts multiples de la fenêtre de congestion dans le même cycle sont obtenues. Puis, nous proposons un cadre analytique pour le choix optimal de la taille du buffer du routeur. Nous formulons le problème du choix optimal de la taille du buffer du routeur comme problème d'optimisation multi-critère, dans lequel la fonction de Lagrange correspond à une combinaison linéaire du taux moyen de transmission et le délai moyen dans le buffer. La solution au problème d'optimisation fournit davantage d'évidence que la taille du buffer du routeur doit être réduite en présence de l'agrégation du trafic. Nos résultats analytiques sont confirmés par des simulations effectuées avec Simulink et le simulateur NS.

Mots-clés : le modèle de TCP/IP, choix optimal de la taille du buffer, modèles hybrides déterministes, l'optimisation multi-critère

1 Introduction

Most traffic in the Internet is governed by TCP/IP (Transmission Control Protocol and Internet Protocol) [1, 14]. Data packets of an Internet connection travel from a source node to a destination node via a series of routers. Some routers, particularly edge routers, experience periods of congestion when packets spend a non-negligible time waiting in the router buffers to be transmitted over the next hop. TCP protocol tries to adjust the sending rate of a source to match the available bandwidth along the path. During the principle Congestion Avoidance phase the current TCP New Reno version uses AIMD (Additive Increase Multiplicative Decrease) binary feedback congestion control scheme. In the absence of congestion signals from the network TCP increases congestion window linearly in time, and upon the reception of a congestion signal TCP reduces the congestion window by a multiplicative factor. Congestion signals can be either packet losses or ECN (Explicit Congestion Notifications) [21]. At the present state of the Internet, nearly all congestion signals are generated by packet losses. Packets can be dropped either when the router buffer is full or when AQM (Active Queue Management) scheme is employed [10]. Given an ambiguity in the choice of the AQM parameters [7, 16], so far AQM is rarely used in practice. On the other hand, in the basic Drop Tail routers, the buffer size is the only one parameter to tune apart of the router capacity. In fact, the buffer size is one of few parameters of the TCP/IP network that can be managed by network operators. This makes the choice of the router buffer size a very important problem in the TCP/IP network design.

The paper is composed of two principle parts. In the first part (Sections 2-5) we analyze the interaction between the AIMD congestion control and the bottleneck router with Drop Tail buffer. This interaction can be adequately described by hybrid modeling approach. There are several hybrid models of the interaction between TCP and the bottleneck router [4, 6, 13]. Here we analyze the model of [13]. To our opinion, this model takes into account all essential details of TCP and at the same time leads to a tractable analysis. We show that the system always converges to a limiting behavior. In particular, we demonstrate that two different limiting regimes can coexist and the convergence to one or to the other depends on the initial conditions. Then, we provide necessary and sufficient conditions for the absence of subsequent packet losses. The absence of subsequent packet losses benefits the TCP performance as well as the quality of service for end users. We note that in [13] there is no characterization of limiting regimes. Furthermore, in [13] only a sufficient condition for the absence of multiple jumps was obtained and the sufficient condition of [13] is loose for some values of the decrease factor.

In the second part of the paper (Sections 6-7) we study the optimal choice of the buffer size in the bottleneck routers. There are some empirical rules for the choice of the router buffer size. The first proposed rule of thumb for the choice of the router buffer size was to choose the buffer size equal to the BDP (Bandwidth-Delay Product) of the outgoing link [23]. This recommendation is based on very approximative considerations and it can be justified only when a router is saturated with a single long-lived TCP connection. The next apparent question to ask was how one should set the buffer size in the case of several competing TCP connections. In [5] it was observed that the utilization of a link improves very fast with

the increase of the buffer size until a certain threshold value. After that threshold value the further increase of the buffer size does not improve the link utilization but increases the queueing delay. Then, two contradictory guidelines for the choice of the buffer size have been proposed. In [17] a connection-proportional buffer size allocation is proposed, whereas in [3] it was suggested that the buffer size should be set to the BDP of the outgoing link divided by the square root of the number of TCP connections. A rationale for the former recommendation is that in order to avoid a high loss rate the buffer must accommodate at least few packets from each connection. And a rationale for the latter recommendation is based on the reduction of the synchronization of TCP connections when the number of connections increases. Then, [3, 17] were followed by two works [8, 11] which try to reconcile these two contradictory approaches. In particular, the authors of [8] recommend to follow the rule of [3] for a relatively small number of long-lived connections and, when the number of long-lived bottlenecked connections is large, to switch to the connection-proportional allocation. One of the main conclusions of [11] is that there are no clear criteria for the optimization of the buffer size. Then, the author of [11] proposed a general avenue for research on the router buffer sizing: "Find the link buffer size that accommodates both TCP and UDP traffic." We note that UDP (User Datagram Protocol) [20] does not use any congestion control and reliable retransmission and it is mostly employed for delay sensitive applications such as Internet Telephony. We refer the interested reader to [24] and references therein for more information on the problem of optimal choice of buffer size.

All the above mentioned works on the router buffer sizing are based on quite rough approximations and strictly speaking do not take into account the feedback nature of TCP protocol. Here we propose a mathematically solid framework to analyze the interaction of TCP with the finite buffer of an IP router. In particular, we state a criterion for the choice of the optimal buffer size in a mathematical form. Our optimization criterion can be considered as a mathematical formalization of the lingual criterion proposed in [11]. Furthermore, the Pareto set obtained for our model allows us to dimension the IP router buffer size to accommodate both data traffic and real time traffic.

All proofs are provided in the Appendix.

2   Mathematical model

The window based binary feedback congestion control can be described by two functions $f(w)$ and $G(w)$. Function $f(w)$ defines the increase profile of the congestion window and function $G(w)$ represents the reduction of the congestion window upon the reception of congestion notification. Namely, in the absence of congestion notification the evolution of congestion window $w(t)$ is described by the differential equation

$$\frac{dw}{dt} = \frac{f(w)}{T + x(t)/\mu}, \tag{1}$$

where T is the two way propagation delay, $x(t)$ is the amount of data in the bottleneck queue and μ is the capacity of the bottleneck router. We note that $T + x(t)/\mu$ corresponds

to the Round Trip Time (RTT) when the amount of the enqueued data in the bottleneck router buffer is $x(t)$ at time moment t. Thus, function $f(w)$ determines the increase of the congestion window per one Round Trip Time. The sending rate $\lambda(t)$ of the window based congestion control is given by

$$\lambda(t) = \frac{w(t)}{T + x(t)/\mu}. \tag{2}$$

We would like to emphasize that here the time parameter t corresponds to the local time observed at the router.

We study a Drop Tail buffer with size B. If $x(t) < B$, the congestion window w increases according to (1). When x reaches B at time t^*, i.e. $x(t^*) = B$, the buffer starts to overflow. The overflow of the buffer will be noticed by the sender only after the time delay $\delta = T + B/\mu$. Upon the reception of the congestion signal at time $t^* + \delta$, the congestion window is reduced according to

$$w(t^* + \delta + 0) = G(w(t^* + \delta - 0)). \tag{3}$$

As we shall see below, w (resp., λ) can represent either a congestion window (resp., sending rate) for a single TCP connection or a total window (resp., total rate) of several TCP connections.

Consider n long-lived AIMD TCP connections that share a bottleneck router. Denote by $w_i(t)$ the instantaneous congestion window of connection $i = 1, ..., n$ at time $t \in [0, \infty)$. In the case of the AIMD congestion control, if $x < B$ the evolution of the congestion window w_i is given by differential equation (1) with $f(w) = m_i = const$. If we restrict ourselves to the symmetric case $T_i = T = const$ and $m_i = m_0 = const$, the sum of all congestion windows $w(t) = \sum_{i=1}^{n} w_i(t)$ also satisfies differential equation (1) with $f(w) = m$, where $m = nm_0$. Namely, we have

$$\frac{dw}{dt} = \frac{m}{T + x(t)/\mu}, \tag{4}$$

$$\frac{dx}{dt} = \begin{cases} \lambda(t) - \mu, & \text{if } 0 < x(t) < B, \text{ or } x(t) = 0 \text{ and } \lambda(t) \geq \mu, \text{ or } x(t) = B \text{ and } \lambda(t) \leq \mu; \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

where $\lambda(t)$ is given by (2). And if $x(t^*) = B$ at some time moment t^*, the congestion window is decreased multiplicatively after the information propagation delay $\delta = T + B/\mu$ as follows:

$$w(t^* + \delta + 0) = \beta^k w(t^* + \delta - 0), \tag{6}$$

and consequently, $G(w) = \beta^k w$ for the AIMD case. Usually, $k = 1$, but sometimes it is necessary to send several congestion signals in order to reduce the sending rate below the transmission capacity of the bottleneck router.

Since we consider the case of equal propagation delays, the synchronization phenomenon takes place [10], and consequently, the total sending rate is also reduced by the factor β^k. For instance, in TCP New Reno version the reduction factor β is equal to one half.

Let us make the change of time scale according to

$$ds \overset{\triangle}{=} \frac{dt}{T + x(t)/\mu}.$$

and the change of variables:

$$v \overset{\triangle}{=} w/m, \quad y \overset{\triangle}{=} x/m.$$

The new time s can be viewed as a counter for Round Trip Times. Now the dynamics of the system between the jumps is described by equations

$$\frac{dv}{ds} = 1, \tag{7}$$

$$\frac{dy}{ds} = \begin{cases} v(t) - y(t) - q, & \text{if } 0 < y(t) < b, \text{ or } y(t) = 0 \text{ and } v(t) \geq q, \text{ or } y(t) = b \text{ and } v(t) \leq q + b; \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

where $q = \mu T/m$ is the maximal number of packets that can be fit in the pipe, in other words Bandwidth-Delay Product (BDP) in packets, and $b = B/m$ is the maximal number of packets that can be fit in the router buffer. Let s^* be the moment in the new time scale when component y reaches value b. Then, equation (6) is transformed to

$$v(s^* + 1 + 0) = \beta^k v(s^* + 1 - 0), \tag{9}$$

where $k = \min\{i : \beta^i v(s^* + 1 - 0) < b + q\}$.

Remark 1 *Because of the delay in the information propagation, the congestion window is reduced after the delay $\delta = T + B/\mu$ in the original time scale, or, equivalently, after 1 time unit in the new time scale s. The value of k is such that, after sending k congestion signals, the amount of data x (and y) starts to decrease.*

3 Convergence of the system trajectories

The dynamics is defined by three parameters β, q, and b, and the system trajectory remains in the region $\Omega = \{0 \leq y \leq b, \ v > 0\}$, provided the initial condition is there.

Suppose a trajectory starts at $s = 0$ from initial condition $y_0 = b$, $\beta(q+b) \leq v_0 < b+q$,[1] and s^* is the first moment when $y(s^*) = b$. Let $v_1 = v(s^* + 1 + 0)$. We introduce mapping φ such that $v_1 \overset{\triangle}{=} \varphi(v_0)$. Consider the iterations $v_{i+1} \overset{\triangle}{=} \varphi(v_i)$, $i = 0, 1, ...$

Theorem 1 *There exists $\lim_{i \to \infty} v_i = V(v_0)$ with*

$$V(v) = \begin{cases} V_1, & \text{if } v \in [\beta(q+b), d]; \\ V_2, & \text{if } v \in (d, q+b), \end{cases} \tag{10}$$

for some constant d. In particular, one of the above intervals can be empty.

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[1] Initial conditions outside the region $[\beta(q + b), q + b)$ are of no interest because, after the very first (multiple) jump we have $v(s^* + 1 + 0) \in [\beta(q + b), q + b)$.

Definition 1 *Suppose the trajectory starting at $s = 0$ from initial condition $y_0 = b$, $v_0 < b + q$ reaches the same point, for the first time, at some time moment $S \geq 1$. Then this finite trajectory is called a* cycle. *A cycle with component y remaining zero for a positive time interval is called* clipped *(see Figure 1). If a cycle touches the axis $y = 0$ only at a single point, we call such cycle* critical *(see Figure 2).*

Corollary 1 (from Theorem 1) *Any cycle has a single time moment, when a (multiple) jump occurs.*

The number k of instant jumps of component v is called a *cycle order*. We call such cycles k-cycles for brevity. If one of the intervals in (10) is empty then only a single cycle exists (Figure 1). Otherwise, two cycles exist simultaneously (Figure 3); their orders are two subsequent positive integers. According to Theorem 1, which cycle is realized depends on the initial conditions.

4   Properties of cycles

In this section, we characterize the shape of cycles. In other words, for given parameters β, q and b, we would like to know if the limit cycles of the system trajectories are clipped or unclipped and what orders the cycles have. For fixed values of β and q, we define the following quantities:

$$N \triangleq \min\left\{ i \geq 1 : \ \frac{\beta^i}{1 - \beta^i} < q \right\}; \tag{11}$$

$$D \triangleq \ln(1 - \beta^N) + \frac{2\beta^N}{1 - \beta^N}; \tag{12}$$

$$C \triangleq -\ln(1 - \beta^N) - \beta^N; \tag{13}$$

θ_k is the single positive solution to equation

$$\ln \frac{\theta}{1 - e^{-\theta}} + \frac{\beta^k \theta}{1 - \beta^k} = q - \frac{\beta^k}{1 - \beta^k}, \qquad k = N, N+1; \tag{14}$$

$$b_{0,k} \triangleq \frac{\theta_k}{1 - e^{-\theta_k}} - \ln \frac{\theta_k}{1 - e^{-\theta_k}} - 1. \tag{15}$$

Then, we define the set of quantities which do not depend on q:
τ_k is the single positive solution to equation

$$\frac{\tau}{1 + \frac{\beta^{k-1} - \beta^k}{1 - \beta^k}(\tau + 1)} = 1 - e^{-\tau}, \quad k = 2, 3, \dots \tag{16}$$

$$A_k^* \triangleq \frac{\beta^{k-1}(\tau_k + 1)}{1 - \beta^k}; \tag{17}$$

$$q_k^* \overset{\triangle}{=} \frac{\beta^k}{1 - \beta^k}(\tau_k + 1) + \ln \frac{\tau_k}{1 - e^{-\tau_k}}; \qquad (18)$$

It is convenient to put τ_1, A_1^* and q_1^* equal to $+\infty$. Finally, in case $q \leq D$ one has to solve equation

$$e^{-r} + r - 1 = \beta^N(q + r + 1) - q. \qquad (19)$$

It has no more than two positive solutions $\underline{r} \leq \bar{r}$ which define

$$\underline{b} \overset{\triangle}{=} e^{-\underline{r}} + \underline{r} - 1; \quad \bar{b} \overset{\triangle}{=} e^{-\bar{r}} + \bar{r} - 1, \qquad (20)$$

Note that $\underline{b} \leq \bar{b}$. If $q \leq q_{N+1}^*$ then $q \leq D$ and $\bar{b} \geq A_{N+1}^* - q$.

We note that all the above defined quantities do not depend on b. Thus, from now on we assume that β and q are fixed and we are going to describe what kind of cycles exist for different values of b. In other words, we study what effect the router buffer size has on the limiting behavior of TCP/IP. There are three cases:

Case $A_{N+1}^* < q$.

If $b \in \left[0, \frac{\beta^{N-1}}{1-\beta^{N-1}} - q\right]$ then only the cycle of order N exists. In case $N = 1$, we put $\frac{\beta^0}{1-\beta^0} = +\infty$ for generality.

Suppose $N > 1$. Then for $b \in \left(\frac{\beta^{N-1}}{1-\beta^{N-1}} - q, A_N^* - q\right]$ two cycles, of orders N and $N - 1$ exist simultaneously. For $b \in \left(A_N^* - q, \frac{\beta^{N-2}}{1-\beta^{N-2}} - q\right]$, there exists only a single cycle of order $N - 1$. And so on; for $b > A_2^* - q$, only 1-cycle exists (see Figure 13).

The N-cycle is clipped for $b \in [0, b_{0,N})$. Cycles of lower orders are unclipped for all values of b, if they exist.
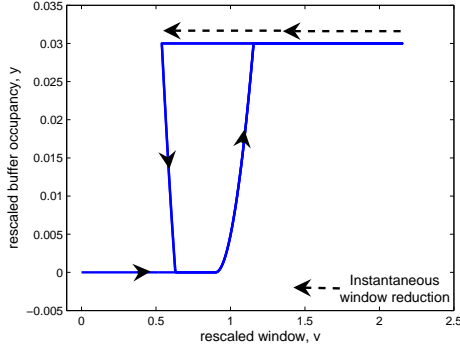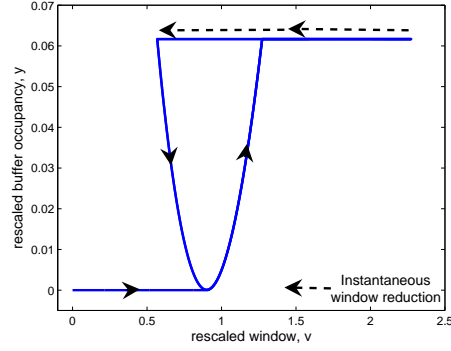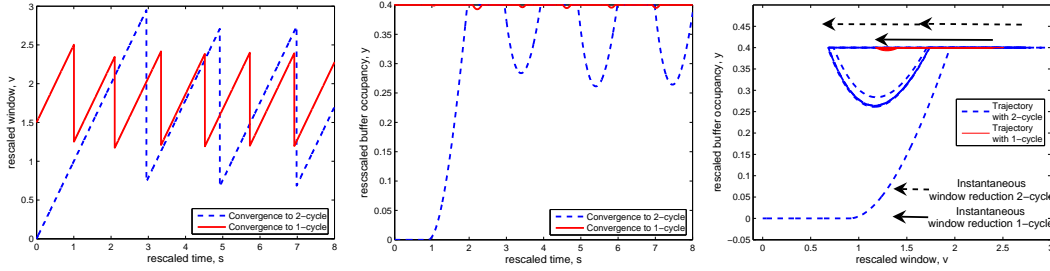
The N-cycle touches the v-axis at a single point iff $b = b_{0,N}$. Thus, if $b = b_{0,N}$ there exists a critical N-cycle. No critical cycles of lower orders exist.

Example 1 *Let us illustrate this with a numerical example. If we take $q = 0.9$ and $\beta = 1/2$ then $N = 2$, $A_2^* = 1.4965$, $A_3^* = 0.3910$. If $b \in [0, 0.1]$ we have only 2-cycles; if $b \in (0.1, 0.5965]$ we have 1-cycles and 2-cycles (see Figure 3); and if $b > 0.5965$ we have only 1-cycles. For each $b < b_{0,2} = 0.0617$, there exists only a clipped 2-cycle (see Figure 1). As one can see on Figure 2, when $b = b_{0,2} = 0.0617$, the 2-cycle becomes critical. All figures for this example have been plotted with MATLAB Simulink.*

Case $q \leq q_{N+1}^*$.

If $b \in [0, \underline{b})$, then only the N-cycle exists. If $b \in [\underline{b}, A_{N+1}^* - q]$, then two cycles of orders N and $N + 1$ exist simultaneously. For $b \in (A_{N+1}^* - q, \frac{\beta^{N-1}}{1-\beta^{N-1}} - q]$, again, only the N-cycle exists.

The N-cycle is clipped for $b \in [0, b_{0,N})$; the $(N + 1)$-cycle is clipped for $b \in [\underline{b}, b_{0,N+1})$. These cycles become critical at $b = b_{0,N}$ and $b = b_{0,N+1}$, respectively. Cycles of lower orders

Figure 1: Clipped 2-cycle. Case $A_2^* < q$.

Figure 2: Critical 2-cycle. Case $A_2^* < q$.

Figure 3: Co-existence of 1-cycle and 2-cycle. Case $A_2^* < q$.

are unclipped for all values of b, if they exist. If $N > 1$ then, similarly to the case $A_{N+1}^* < q$, the order of the cycle decreases as b increases above $\frac{\beta^{N-1}}{1-\beta^{N-1}} - q$ (see Figure 13).

Case $q_{N+1}^* < q \le A_{N+1}^*$.

If $C \le A_{N+1}^* - q$ [2] and $q \le D$, then everything is similar to the case $q \le q_{N+1}^*$. The difference is that the $(N+1)$-cycle is clipped and cannot be critical; it exists simultaneously with the N-cycle for $b \in [\underline{b}, \bar{b}]$. If $b \in \left(\bar{b}, \frac{\beta^{N-1}}{1-\beta^{N-1}} - q\right]$, only the N-cycle exists. The latter interval is non-empty.

If $C > A_{N+1}^* - q$ or $D < q$, then everything is exactly as in case $A_{N+1}^* < q$.

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[2] Actually C cannot be equal to $A_{N+1}^* - q$.

5   Conditions for the absence of multiple jumps

The regime with multiple jumps is not desirable. The multiple jump corresponds to the lost of more than one packet in a single congestion window. Subsequent packet losses can force TCP to switch from the Congestion Avoidance TCP phase to the Slow Start phase and lead to lengthy timeouts. Furthermore, the absence of subsequent packet losses is beneficial not only for the TCP performance but also for the quality of service provided to the end users. In the next theorem we provide necessary and sufficient conditions for the absence of multiple jumps, namely, we characterize all possible cases when only a single cycle of order 1 exists.

Theorem 2 *The following mutually exclusive conditions fully characterise all possible cases when only a single cycle of order 1 exists:*

(a) $\frac{\beta}{1-\beta} \geq q$ *and* $b + q > A_2^*$;

(b) $A_2^* < q$ *(b can be arbitrary)*;

(c) $\frac{\beta}{1-\beta} < q \leq q_2^*$ *and* $b \notin [\underline{b}, A_2^* - q]$;

(d) $\max\left\{ \frac{\beta}{1-\beta}, \ q_2^* \right\} < q \leq A_2^* - C$, $q \leq D$ *and* $b \notin [\underline{b}, \bar{b}]$;

(e) $\max\left\{ \frac{\beta}{1-\beta}, \ q_2^* \right\} < q \leq A_2^* - C$, $q > D$ *(b can be arbitrary)*;

(f) $\max\left\{ \frac{\beta}{1-\beta}, \ q_2^*, \ A_2^* - C \right\} < q \leq A_2^*$ *(b can be arbitrary)*.

In the following corollary we provide a simple sufficient condition for absence of multiple jumps.

Corollary 2 *Condition* $b + q > A_2^*$ *is sufficient for the absence of cycles of orders* $k > 1$. *(See Figure 13 and Corollary 6.)*

Recall that A_2^* depends only on β. In particular, if $\beta = 1/2$, $A_2^* = 1.4965$.

We would like to note that the above sufficient condition is tighter than the sufficient condition for the absence of multiple jumps provided in [13]: $b + q > 2\beta/(1 - \beta)$. To compare these two conditions, we plot $A_2^*(\beta)$ and $2\beta/(1 - \beta)$ in Figure 4 and the difference $2\beta/(1 - \beta) - A_2^*(\beta)$ in Figure 5. Strictly speaking we have

Proposition 1 *The difference* $\delta \overset{\triangle}{=} \frac{2\beta}{1-\beta} - A_2^*$ *is always positive and* $\lim_{\beta \to 1} \delta = +\infty$.

Nevertheless, the simple sufficient condition of [13] appears to be quite good except for values of β that are too close to one.
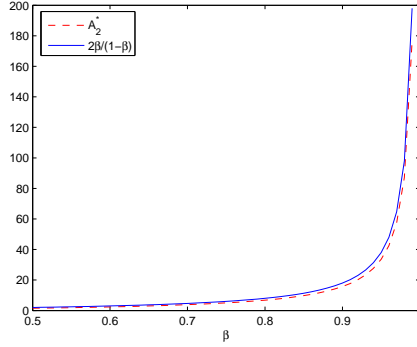
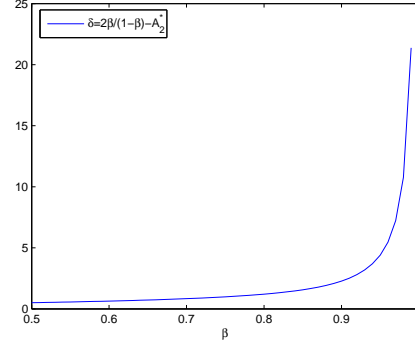Figure 4: Comparison of sufficient conditions for the absence of multiple jumps.

Figure 5: The value of $2\beta/(1-\beta) - A_2^*(\beta)$.

6 Pareto set for optimal buffer sizing

Let us study what effect has the choice of the buffer size on the performance of TCP. In particular, we are interested in optimal buffer sizing. Towards this goal, let us formulate the performance criteria. On one hand, we are interested to obtain as large goodput as possible. That is, we are interested to maximize the average goodput

$$\bar{g} = \lim_{t \to \infty} \frac{1}{t} \int_0^t g(s)ds,$$

where the instantaneous goodput $g(t)$ is defined by

$$g(t) = \begin{cases} \lambda(t), & \text{if} \quad x(t) < B, \\ \mu, & \text{if} \quad x(t) = B. \end{cases}$$

On the other hand, we are interested to make the delay of data in the buffer as small as possible. That is, we are also interested to minimize the average amount of data in the buffer

$$\bar{x} = \lim_{t \to \infty} \frac{1}{t} \int_0^t x(s)ds.$$

Clearly, these two goals are contradictory. In fact, here we face a typical example of multi-criteria optimization. A standard approach to it is to consider the optimization of one criterion under constraints for the other criteria (see e.g., [19]). Namely, we would like to maximize the goodput given that the average amount of data in the buffer does not exceed a certain value

$$\max\{\bar{g} : \bar{x} \le \bar{x}_*\}. \tag{21}$$

Or we would like to minimize the average delay given that the average goodput is not less than a certain value

$$f \min\{\bar{x} : \bar{g} \geq \bar{g}_*\}. \tag{22}$$

The solution to the above constrained optimization problems can be obtained from the Pareto set. As is known, see e.g. [19], the Pareto set can be constructed by solving the optimization problem

$$\max \left\{ \lim_{t \to \infty} \frac{1}{t} \int_0^t c_1 g(s) - c_2 x(s) ds \right\}. \tag{23}$$

To be more precise, the Pareto Set is formed by the pairs of objectives $(\bar{g}, \bar{x})$ that solve (23) for different $(c_1, c_2) \in R_+^2$. An example of Pareto set is given in Figure 6. Each point of the Pareto set corresponds to a solution of optimization problem (23) for some choice of c_1 and c_2. Once we obtain the Pareto set, it is very easy to deduce solution of problems (21) and (22). For instance, if one wants that the utilization of the bottleneck router will be not less than, say, 95%, one has to be ready to accept the delays that are equal or greater than x_*.
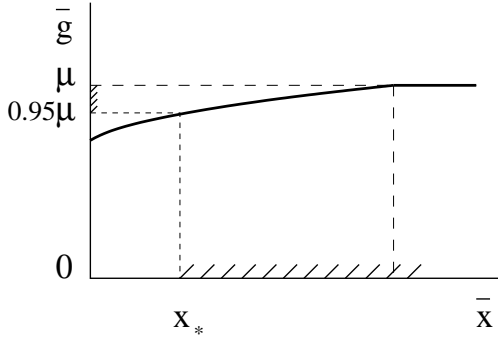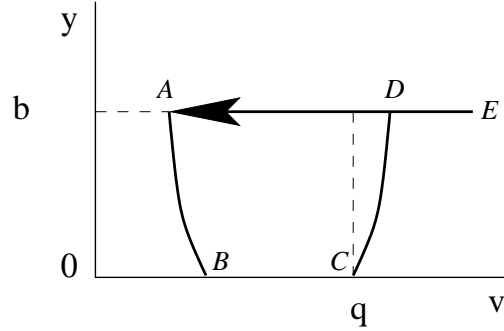

Figure 6: Pareto set.

Figure 7: Phases of the clipped cycle.

All three optimization problems (21), (22) and (23) can be regarded as mathematical formulation of the lingual criterion "find the link buffer size that accommodates both TCP and UDP traffic" given in [11]. Since UDP traffic does not contribute much in terms of the load, for the design of IP routers one can use for instance optimization problem (21) where the delay constraint is imposed by the UDP traffic.

We note that here we deal with the optimal impulse control problem of a deterministic system with long-run average optimality criterion. To the best of our knowledge there are no available results on such type of problems in the literature. In principle, the control policy in our model can depend on the current values of x and λ. In practice, however, all currently implemented buffer management schemes (e.g., AQM, DropTail) send congestion signals based only on the state of the buffer. Thus, we also limit ourselves to the case when the control depends only on the amount of data in the buffer. Furthermore, we restrict the

control action only to the choice of the buffer size. Thus, the control signal is only sent at the moment when the buffer gets full.

The following theorem provides expressions for the average sending rate, goodput and queue size under condition $q > A_2^*$, which guarantees the absence of multiple jumps for any value of the buffer size. Remember that A_2^* depends only on β (see (16),(17)). In particular, the expressions allow us to plot the Pareto set parameterized by the buffer size.

Theorem 3 *Let the condition $\mu T/m > A_2^*$ be satisfied. Then, for $B \in [0, mb_{0,1}]$ the average sending rate, goodput and buffer occupancy are given by*

$$\bar{\lambda} = \frac{m(1 - \beta^2)}{2T_{cycle}} \left(1 + \frac{\mu T}{m} + S_{CD} \right)^2,$$

$$\bar{g} = \frac{m}{T_{cycle}} \left[\frac{1}{2} \left(\frac{\mu T}{m} + S_{CD} \right)^2 - \frac{\beta^2}{2} \left(1 + \frac{\mu T}{m} + S_{CD} \right)^2 + \frac{\mu T + B}{m} \right],$$

$$\bar{x} = \frac{1}{T_{cycle}} \left[mT \left(\int_0^{S_{AB}} y_{AB}(s)ds + \int_0^{S_{CD}} y_{CD}(u)du \right) \right.$$

$$\left. + \frac{m^2}{\mu} \left(\int_0^{S_{AB}} y_{AB}^2(s)ds + \int_0^{S_{CD}} y_{CD}^2(u)du + \frac{B(\mu T + B)}{m^2} \right) \right],$$

respectively, where T_{cycle} is the cycle duration given by

$$T_{cycle} = (1 - \beta)(1 + \frac{\mu T}{m} + S_{CD})T + \frac{B}{\mu} + \frac{m}{\mu} \left(\int_0^{S_{AB}} y_{AB}(s)ds + \int_0^{S_{CD}} y_{CD}(u)du \right),$$

with

$$y_{CD}(u) = e^{-u} + (u - 1),$$

$$y_{AB}(s) = [\frac{B}{m} + (1 - \beta)(1 + \frac{\mu T}{m}) - \beta S_{CD}]e^{-s} + (s - 1) + \beta(S_{CD} + 1) - (1 - \beta)\frac{\mu T}{m},$$

where S_{CD} and S_{AB} are the solutions of the equations

$$e^{-S_{CD}} + S_{CD} - 1 = \frac{B}{m},$$

$$\left[\frac{B}{m} - \beta S_{CD} + (1 - \beta)(1 + \frac{\mu T}{m}) \right] e^{-S_{AB}} + S_{AB} + \beta S_{CD} - (1 - \beta)(1 + \frac{\mu T}{m}) = 0.$$

For $B \in (mb_{0,1}, \infty)$, we have

$$\bar{\lambda} = \frac{m}{2T_{cycle}} \frac{1 + \beta}{1 - \beta}(s_1 + 1)^2,$$

$$\bar{g} = \mu,$$

$$\bar{x} = \frac{1}{T_{cycle}} \left[mT \int_0^{s_1} y(s)ds + \frac{m^2}{\mu} \left(\int_0^{s_1} y^2(s)ds + \frac{B(\mu T + B)}{m^2} \right) \right],$$

where

$$T_{cycle} = T(s_1 + 1) + \frac{m}{\mu} \left(\int_0^{s_1} y(s)ds + \frac{B}{m} \right)$$

with

$$y(s) = \left[1 + \frac{\mu T + B}{m} - v_0 \right] e^{-s} + (s - 1) + v_0 - \frac{\mu T}{m},$$

where v_0 and s_1 are defined by (28) and (29) with $k = 1$.

Example 2 *Let us illustrate the Pareto set for a benchmark example of the TCP/IP network created with the help of NS-2 simulator [18]. The network consists of a single bottleneck link of capacity $\mu = 10Mbps$ which is shared by n long-lived TCP connections. The propagation delay for each connection is $T = 0.24s$ and $\beta = 1/2$. The packet size is $4000bits$. Thus, we have that $m_0 = 4000bits$ as well. In Figure 8 we plot the Pareto set for $n = 10$ (and $m = nm_0 = 40,000$) using the formulae of Theorem 3 and measurements obtained from NS simulations. As one can see, two curves match well. In Figure 9, again using the formulae of Theorem 3, we plot the average goodput and the average sending rate as functions of the buffer size for $n = 60$.*

We note that $\bar{g} \leq \mu$ always, but the average sending rate $\bar{\lambda}$ can exceed the router capacity μ (see Figure 9). Nevertheless, as the next Proposition 2 states, the difference between the average sending rate and the router capacity goes to zero as B increases. In particular, this means that when the Drop Tail router is used, the rate of lost (and then retransmitted) information eventually diminishes to zero as the buffer size increases.

Proposition 2 *When $B \to \infty$, the difference $\Delta = \bar{\lambda} - \mu$ approaches zero from above.*

7   Minimal buffer size for the full system utilization

In the case of multiple TCP connections competing for resource of the bottleneck router we have $m = nm_0$. Here n is the number of competing TCP connections. Let us study how the minimal buffer size for the full system utilization, $B_{0,N}$, depends on n or, equivalently, on m. $B_{0,N}$ is the buffer size corresponding to scenario when the Pareto set touches the level μ (see Figure 6). It corresponds also to the critical cycle of minimal order.

Proposition 3 *(a) For a fixed N, the value of $B_{0,N} = mb_{0,N}$ decreases as m increases.*
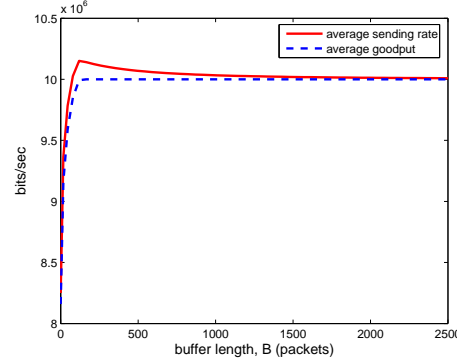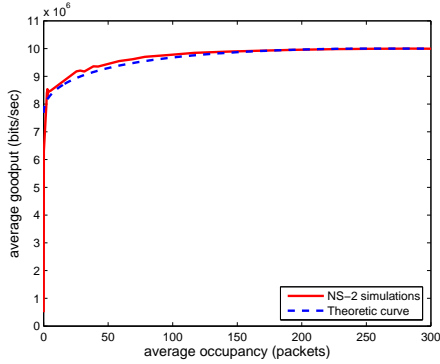*(b) The value of $B_{0,N}$ increases as N increases.*

Figure 8: Pareto set: Numeric calculations and NS-2 simulations.

Figure 9: Non-monotonicity of the average sending rate.

Corollary 3 *The buffer size $B_{0,N}$ of the minimal order critical cycle is a piece-wise differentiable function of m, decreasing on the intervals $[m_i, m_{i+1})$;*

$$\lim_{m \to m_{i+1} - 0} B_{0,N}(m) < B_{0,N}(m_{i+1}), \quad i = 0, 1, 2, \ldots$$

Here $m_i \triangleq \mu T(1 - \beta^i)/\beta^i$; the value of N equals $i+1$ on the interval $[m_i, m_{i+1})$ (see (11)).

Moreover, $\lim_{m \to m_N - 0} B_{0,N} = 0$, $\lim_{m \to m_N - 0} \frac{dB_{0,N}}{dm} = 0$, $\lim_{m \to 0+} B_{0,1} = \mu T(1 - \beta)/\beta$, $\lim_{N \to \infty} m_{N-1} B_{0,N}(m_{N-1}) = 0.5(\mu T(1 - \beta))^2$ and hence $\lim_{m \to \infty} B_{0,N} = 0$.

Example 2(cntd.) *In Figure 10 we plot the buffer size $B_{0,N}$ of the minimal order critical cycle and the curve $f(m) = (1 - \beta)^2(\mu T)^2/(2m)$ for $\mu T = 2.4 \times 10^6 bits$ (600packets). The curve $f(m)$ indeed approaches fast the local maxima of $B_{0,N}$ as m increases. In Figure 11 we make a zoom on the interval with smaller values of m. As one can see, when m goes to zero, the value of $B_{0,N}$ approaches 600packets, which is the BDP in this network example.*

We note that by Corollary 3 for small values of m the minimal buffer size for the full system utilization is approximately equal to μT, BDP of the bottleneck link. This is in agreement with the empirical conclusion of [23]. In [3] the authors suggested that the minimal buffer size for the full system utilization should decrease as $(\mu T)/\sqrt{n}$ as the number of connections n increases. We note that the authors of [3] have assumed that the competing TCP connections are not synchronized. That is, only a single connection reduces its congestion window when the buffer becomes full. In our model we assume full synchronization of competing TCP connections. Namely, when the buffer is full, all connections simultaneously reduce their congestion windows. We expect that the situation in real networks is in between these two extremes. And thus, the model of [3] provides an upper bound and our model provides a lower bound. Furthermore, it was believed previously that if the competing TCP connections are synchronized, one has to provide BDP of buffering to guarantee

the full system utilization. From Figure 11 one can see that the minimal buffer requirement decreases with increasing m (or, equivalently, with increasing n) even in the case of complete synchronization. Finally, we would like to mention that the value of $B_{0,N}$ is non-monotonous with respect to m, even though it eventually decreases to zero (see Figure 10). Curiously enough, the experiments of [24] with the router, running FreeBSD dummynet software, have also shown the non-monotonous behavior of the minimal buffer requirement in the case of synchronized connections (see Figure 1 in [24]).
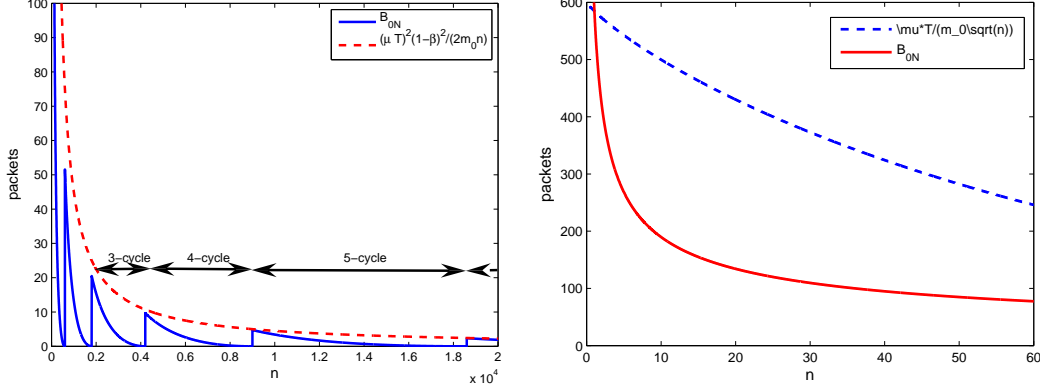

Figure 10: The minimal buffer size for the full system utilization.

Figure 11: The minimal buffer size for the full system utilization (zoom).

8   Conclusions

In this paper we have studied the interaction between AIMD Congestion Control and a bottleneck router with Drop Tail buffer. We have used the hybrid modeling approach. It is demonstrated that the system always converges to a cyclic behavior. The limit cycles have been fully characterized. In particular, we have obtained necessary and sufficient condition for the absence of cycles with multiple jumps and a simple but tight sufficient condition. Then, we have formulated the problem of choosing the buffer size of routers in the Internet as a multi-criteria optimization problem. In agreement with previous works, our model suggests that as the number of long-lived TCP connections sharing the common link increases, the minimal buffer size required to achieve full link utilization decreases. However, in the case of synchronized connections, the decrease is not monotonous and slower than the inverse of the square root of the number of connections. The Pareto set obtained with the help of our model allows us to evaluate the IP router buffer size in order to accommodate real time traffic as well as data traffic. The simulations carried out with the help of Simulink and NS Simulator confirm the qualitative insights drawn from our model. Application of the same

framework to other congestion control mechanisms, such as MIMD, HighSpeed TCP, TCP Westwood appears to be a fruitful direction for future research.

Appendix

Unclipped cycles.

In this and the next subsection, we ignore the requirement that $y \geq 0$. Thus dynamics is described by equations

$$\begin{cases} \frac{dv}{ds} = 1; \\ \frac{dy}{ds} = \begin{cases} v - y - q, & \text{if } y < b, \text{ or} \\ & \quad y = b \text{ and } v \leq A; \\ 0 & \quad \text{otherwise,} \end{cases} \end{cases} \tag{24}$$

where

$$A \overset{\triangle}{=} b + q.$$

The jumps occur according to (7) as before.

Definition 2 *Let $y_0 = b$ and $v_0 < A$ be the initial conditions. A piece of trajectory on the time interval $[0, s^* + 1 + 0]$ is called a pseudo-cycle of order k (see (7)). If $v(s^* + 1 + 0) = v_0$ then the pseudo-cycle is called a k-cycle.*

Later, it will be shown that if a clipped k-cycle exists then the unclipped k-cycle exists, too (Corollary 6). Clearly, (24) has a single solution

$$\begin{cases} v(s) = v_0 + s; \\ y(s) = (1 + q + y_0 - v_0)e^{-s} + s - 1 + v_0 - q. \end{cases} \tag{25}$$

Theorem 4 *An (unclipped) k-cycle exists iff*

$$A \in \left(\frac{\beta^k}{1 - \beta^k}, \ A_k^* \right], \tag{26}$$

where

$$A_k^* \overset{\triangle}{=} \begin{cases} \frac{\beta^{k-1}(\tau_k + 1)}{1 - \beta^k}, & \text{if } k > 1, \\ \infty, & \text{if } k = 1 \end{cases} \tag{27}$$

and, for $k > 1$, τ_k is the single positive solution to (16).

<u>Proof.</u> Obviously, parameters of a k-cycle, v_0 and time interval s_1 can be found from equations

$$y(s_1) = b; \qquad \beta^k v(s_1 + 1) = v_0,$$

which are equivalent to

$$v_0 = \frac{\beta^k(s_1+1)}{1-\beta^k}.$$

(28)

$$1 - e^{-s_1} = \frac{s_1}{1 + A - \frac{\beta^k(s_1+1)}{1-\beta^k}}.$$

(29)

A k-cycle exists iff (29) has a positive solution and v_0 given by (28) satisfies inequality $v_0 \geq \beta A$. (Otherwise, if $v_0 < \beta A$, there is no need to reduce v so many times.) Equation (29) has a positive solution iff

$$1 + A - \frac{\beta^k}{1-\beta^k} > 0 \quad \text{and} \quad \frac{d}{ds}\left[\frac{s}{1 + A - \frac{\beta^k(s+1)}{1-\beta^k}}\right]\Bigg|_{s=0} < 1$$

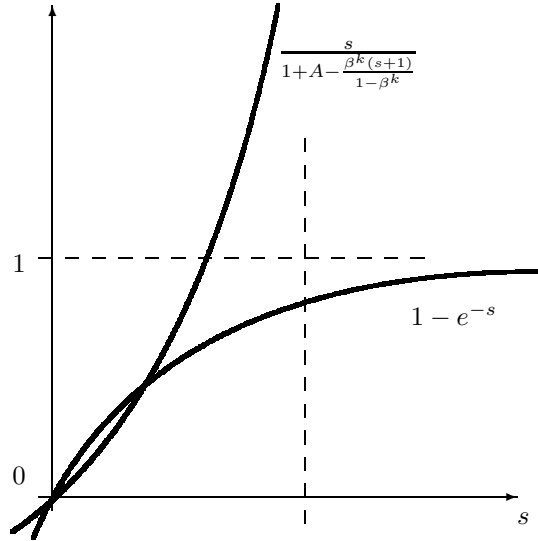Figure 12: Graphical solution to equation (29).

(see Fig.12), or, equivalently, iff

$$\beta^k(1+A) < A.$$

(30)

Put

$$K \stackrel{\triangle}{=} \min\{i \geq 1 : \quad \beta^i < \frac{A}{1+A}\}.$$

(31)

Before proceeding further, we need the following statements.

Lemma 1 *If $v_0 \in [\beta A, A)$ then, starting from v_0, $y_0 = b$, the next instant series of $K + 1$ jumps results in the value $v < A$. Hence the order of any cycle cannot exceed $K + 1$ (and clearly cannot be smaller than K).*

Proof. Suppose $\hat{v}_0 = \beta A$. Then, after the next instant series of $K + 1$ jumps, the value $\hat{v}$ is not smaller than v. To put it differently,

$$v \leq \beta^{K+1}[\beta A + \hat{s} + 1], \tag{32}$$

where $\hat{s}$ solves equation

$$(1 + A - \beta A)e^{-s} + s - 1 + \beta A = A$$

$$\Longleftrightarrow \frac{s}{1 + (1 - \beta)A} - 1 + e^{-s} = 0. \tag{33}$$

If we substitute

$$\tilde{s} \triangleq \frac{A + 1}{\beta} - \beta A - 1 < \frac{A}{\beta^{K+1}} - \beta A - 1$$

into (33) we obtain, using equality $A = \frac{\beta \tilde{s} + \beta - 1}{1 - \beta^2}$:

$$\frac{\tilde{s}}{1 + (1 - \beta)A} - 1 + e^{-\tilde{s}} = \frac{\tilde{s}(1 + \beta)}{\beta(2 + \tilde{s})} - 1 + e^{-\tilde{s}} > \frac{2\tilde{s}}{1 + \tilde{s}} - 1 + e^{-\tilde{s}} > 0.$$

When s increases from zero, the lefthand side of (33) initially decreases from zero and increases thereafter. Hence $\tilde{s} > \hat{s}$ and (32) implies

$$v < \beta^{K+1}[\beta A + \tilde{s} + 1] < \beta^{K+1}[\beta A + \frac{A}{\beta^{K+1}} - \beta A - 1 + 1] = A.$$

∎

Lemma 2 *Suppose $\beta \in (0, 1)$ is fixed and consider function*

$$f_k(A) \triangleq (1 - \beta^k)A - \beta^{k-1}(s_1 + 1), \tag{34}$$

where s_1 solves (29). The domain of f is given by (30). Then
    (a) $\frac{df_k(A)}{dA} > 0$;
    (b) $f_1(A) < 0$ for all $A > \frac{\beta}{1-\beta}$;
    (c) $\forall k > 1$ equation $f_k(A) = 0$ has a single finite solution A_k^* given by (27); A_k^* decreases as k increases.
    (d) $\forall k > 1$ $A_k^* > \frac{\beta^{k-1}}{1-\beta^{k-1}}$; $\forall k > 2$, $A_k^* \leq \frac{\beta^{k-2}}{1-\beta^{k-2}}$.

<u>Proof</u>. (a) According to the rule of implicit differentiation, applied to equation

$$\left(1 + A - \frac{\beta^k(s_1 + 1)}{1 - \beta^k}\right)\left(1 - e^{-s_1}\right) - s_1 = 0,$$

we have

$$\frac{ds_1}{dA} = -\frac{(1 - e^{-s_1})^2(1 - \beta^k)}{e^{-s_1}s_1(1 - \beta^k) - (1 - e^{-s_1})^2\beta^k - (1 - \beta^k)(1 - e^{-s_1})}.$$

The denominator equals

$$-(1 - e^{-s_1} - s_1 e^{-s_1}) - \beta^k(s_1 e^{-s_1} - e^{-s_1} + e^{-2s_1})$$

$$< \beta^k(e^{-s_1} - e^{-2s_1} - s_1 e^{-2s_1}) - (1 - e^{-s_1} - s_1 e^{-s_1})$$

$$= (1 - e^{-s_1} - s_1 e^{-s_1})(\beta e^{-s_1} - 1) < 0;$$

hence $\frac{ds_1}{dA} > 0$ for $s_1 > 0$.

Now

$$\frac{df_k(A)}{dA} = (1 - \beta^k) - \beta^{k-1}\frac{ds_1}{dA}$$

$$= \frac{(1 - \beta^k)[(1 - \beta^k)(s_1 e^{-s_1} - 1 + e^{-s_1}) + \beta^{k-1}(1 - e^{-s_1})^2(1 - \beta)]}{s_1 e^{-s_1}(1 - \beta^k) + (1 - e^{-s_1})(\beta^k e^{-s_1} - 1)}.$$

The denominator is negative (see above). The nominator does not exceed

$$(1 - \beta^k)(1 - \beta)[(s_1 e^{-s_1} - 1 + e^{-s_1})(1 + \beta^{k-1}) + \beta^{k-1}(1 - e^{-s_1})^2]$$

$$= (1 - \beta^k)(1 - \beta)[(s_1 e^{-s_1} - 1 + e^{-s_1}) + \beta^{k-1}e^{-s_1}(s_1 - 1 + e^{-s_1})].$$

The both terms in the latter square bracket are negative for $s_1 > 0$. Hence $\frac{df_k(A)}{dA} > 0$.

(b) It is sufficient to prove that

$$s_1 > S \triangleq A(1 - \beta) - 1,$$

where s_1 solves (29) at $k = 1$.

Case $S < 0$ is trivial, thus assume that $S > 0$. Let us substitute S into the both sides of (29) and estimate the difference:

$$\frac{S}{1 + A - \frac{\beta(S+1)}{1-\beta}} - 1 + e^{-S} = \frac{S}{1 + \frac{S+1}{1-\beta} - \frac{\beta(S+1)}{1-\beta}} - 1 + e^{-S} = e^{-S} - \frac{2}{S+2} < 0,$$

because function $(S + 2)e^{-S}$ decreases from 2 at $S = 0$. To complete this part of the proof, it is sufficient to notice that, on the interval

$$0 < S < \frac{A(1 - \beta)}{\beta} + \frac{1 - 2\beta}{\beta},$$

the righthand side of (29) is smaller than the lefthandside iff $S < s_1$.

(c) The first part is obvious: A_k^* is given by (27), provided equation (16) has a single positive solution. The latter statement folows from the fact that function

$$g(\tau) = (1 - e^{-\tau})(1 + \alpha(\tau + 1))/\tau$$

decreases to $\lim_{\tau \to \infty} g(\tau) = \alpha$, starting from $\lim_{\tau \to 0} g(\tau) = 1 + \alpha$. Here

$$\alpha \stackrel{\triangle}{=} \frac{\beta^{k-1} - \beta^k}{1 - \beta^k}. \tag{35}$$

Indeed,

$$\frac{dg}{d\tau} = \frac{e^{-\tau}[1 + \alpha + \tau(1 + \alpha + \alpha\tau)] - (1 + \alpha)}{\tau^2} < 0$$

in case $\alpha < 1$, and

$$\alpha = \frac{\beta^{k-1}}{1 + \beta + \ldots + \beta^{k-1}} \leq \frac{1}{1 + 1/\beta} < 1/2. \tag{36}$$

Now, look what happens as k increases. Obviously, functions $\frac{\beta^{k-1}}{1-\beta^k} = \frac{\beta^k}{1-\beta^k} \cdot \frac{1}{\beta}$ and $\alpha = \frac{\beta^k}{1-\beta^k}(\frac{1}{\beta} - 1)$ (see (35)) decrease. According to (27) it remains to prove that τ_k given by (16) increases with α. We rewrite (16) as $(1 + \alpha(\tau + 1))(1 - e^{-\tau}) - \tau = 0$. Hence

$$\frac{d\tau_k}{d\alpha} = -\frac{(\tau_k + 1)(1 - e^{-\tau_k})}{\alpha(1 - e^{-\tau_k}) + (1 + \alpha(\tau_k + 1))e^{-\tau_k} - 1} = -\frac{(\tau_k + 1)^2(1 - e^{-\tau_k})^2}{h(\tau_k)},$$

where $h(\tau) = -2 + 3e^{-\tau} - e^{-2\tau} + \tau e^{-\tau} + \tau^2 e^{-\tau}$. (We have substituted $\alpha = \frac{\tau_k - 1 + e^{-\tau_k}}{(1 - e^{-\tau_k})(\tau_k + 1)}$.) We intend to prove that

$$\frac{dh}{d\tau} = -2e^{-\tau} + 2e^{-2\tau} + \tau e^{-\tau} - \tau^2 e^{-\tau} < 0 \tag{37}$$

when $\tau > 0$. Clearly (37) holds for $\tau \geq 1$.

Suppose $\tau \in (0, 1)$. Then

$$\frac{d^2 h}{d\tau^2} = 3e^{-\tau} - 4e^{-2\tau} - 3\tau e^{-\tau} + \tau^2 e^{-\tau} < 3e^{-\tau} - 4e^{-2\tau} - 2\tau e^{-\tau} = e^{-\tau}(3 - 4e^{-\tau} - 2\tau).$$

Expression in the brackets has a negative maximum at $\tau = \ln 2$. Therefore, $\frac{d^2 h}{d\tau^2} < 0$ and $\frac{dh}{d\tau} < 0$. Finally, $h(\tau) < 0$ for all $\tau > 0$, because $h(0) = 0$.

(d) To estimate A_k^* from below, we use statement (a): it is sufficient to establish that $f_k\left(\frac{\beta^{k-1}}{1-\beta^{k-1}}\right) < 0$, ie $s_1 + 1 > \frac{1-\beta^k}{1-\beta^{k-1}} \iff \frac{S}{1 + \frac{\beta^{k-1}}{1-\beta^{k-1}} - \frac{\beta^k(S+1)}{1-\beta^k}} < 1 - e^{-S}$ for $S = \frac{1-\beta^k}{1-\beta^{k-1}} - 1$. (The argument is similar to (b).) But

$$\frac{S}{\frac{1}{1-\beta^{k-1}} - \frac{\beta^k}{1-\beta^k} \cdot \frac{1-\beta^k}{1-\beta^{k-1}}} - 1 + e^{-S} = \frac{S}{S+1} - 1 + e^{-S} = e^{-S} - \frac{1}{1+S} < 0$$

because function $e^{-S}(1 + S)$ decreases from 1 at $S = 0$.

Finally, in case $k > 2$, suppose $A_k^* > \frac{\beta^{k-2}}{1-\beta^{k-2}}$. Then for parameters values β and $A \in \left(\frac{\beta^{k-2}}{1-\beta^{k-2}}, A_k^*\right)$ we have that (30) holds for $k - 2$, $k - 1$, and k and simultaneously $f_k(A) < 0$, $f_{k-1}(A) < 0$, $f_{k-2}(A) < 0$: see (a) and (c). According to the beginning of the proof of Theorem 1, cycles of orders k, $k - 1$, and $k - 2$ exist which contradicts Lemma 1. ■

Now we can easily finish the proof of Theorem 4. Suppose a k-cycle exists. Then, according to (30), $A > \frac{\beta^k}{1-\beta^k}$. Lemma 2 guarantees that

$$A_k^* > \frac{\beta^{k-1}}{1 - \beta^{k-1}} > \frac{\beta^k}{1 - \beta^k},$$

and, as was mentioned earlier, inequality $v_0 \geq \beta A$ must be valid (see (28)), which is equivalent to $A \leq A_k^*$. Finally, if (26) holds then (29) has a positive solution (see (30)) and $v_0 \geq \beta A$; hence a k-cycle exists. ■
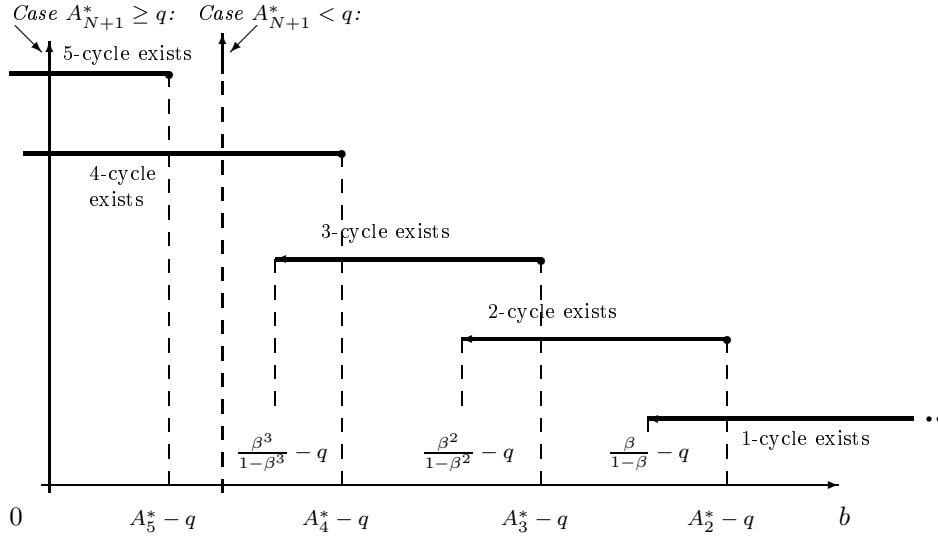

Figure 13: Existence of unclipped cycles; $N = 4$.

Remember that $A = b + q$. Thus, if q is fixed and b increases from 0, unclipped cycles have orders N (see (11)) and, possibly, $N + 1$, if $A_{N+1}^* - q > 0$. Later, as b increases, the order of cycles decreases according to Fig. 13.

Stability of unclipped cycles.

We intend to study the mapping φ introduced just before Theorem 1. Since we study only unclipped cycles, this map is a little different and will be denoted $\tilde{\varphi}$. But firstly we

concentrate on a different mapping:

$$\Phi^k(v_0) = \beta^k(v_0 + s^* + 1)$$

defined for $v_0 \in [\beta A, A]$ under a fixed $k \geq 1$. Here $s^* \stackrel{\triangle}{=} 0$ if $v_0 = A$; in case $v_0 < A$, $s^* > 0$ is the first moment when $y(s^*) = b$ starting from $y(0) = b$, $v(0) = v_0$.

Lemma 3 $\left| \frac{d\Phi^k(v_0)}{dv_0} \right| < \beta^k$ and hence Φ^k is a contraction. Function Φ^k is decreasing.

<u>Proof.</u> Assuming that $v_0 < A$, s^* is a single positive solution to equation

$$(1 + A - v_0)(1 - e^{-s}) - s = 0, \tag{38}$$

hence

$$\frac{ds^*}{dv_0} = \frac{1 - e^{-s^*}}{(1 + A - v_0)e^{-s^*} - 1} = \frac{1 - e^{-s^*}}{\frac{s^*}{1 - e^{-s^*}} \cdot e^{-s^*} - 1}$$

and

$$\frac{d\Phi^k}{dv_0} = \beta^k(1 + \frac{ds^*}{dv_0}) = \beta^k e^{-s^*} \frac{s^* - 1 + e^{-s^*}}{s^* e^{-s^*} + e^{-s^*} - 1} < 0.$$

Finally,

$$e^{-s^*} \frac{s^* - 1 + e^{-s^*}}{s^* e^{-s^*} + e^{-s^*} - 1} + 1 = e^{-s^*} \frac{e^{s^*} - e^{-s^*} - 2s^*}{1 - e^{-s^*} - s^* e^{-s^*}} > 0,$$

because the nominator increases, starting from 0 at $s^* = 0$. Therefore $\frac{d\Phi^k}{dv_0} > (-\beta^k)$. $\blacksquare$

Lemma 4 *(a)* $A \in \left(A_{K+1}^*, \frac{\beta^{K-1}}{1 - \beta^{K-1}} \right]$ *iff* $d < \beta A$, *where* d *is a solution to* $\Phi^K(d) = A$. *Here and below,* $\frac{\beta^{K-1}}{1 - \beta^{K-1}} \stackrel{\triangle}{=} \infty$ *if* $K = 1$; K *is defined by (31).*
In this case, $\forall v_0 \in [\beta A, A)$, *the mapping* $\tilde{\varphi}(v_0)$ *coincides with* $\Phi^K(v_0)$.
(b) If $A \in \left(\frac{\beta^K}{1 - \beta^K}, A_{K+1}^* \right]$ *the following statements hold:*
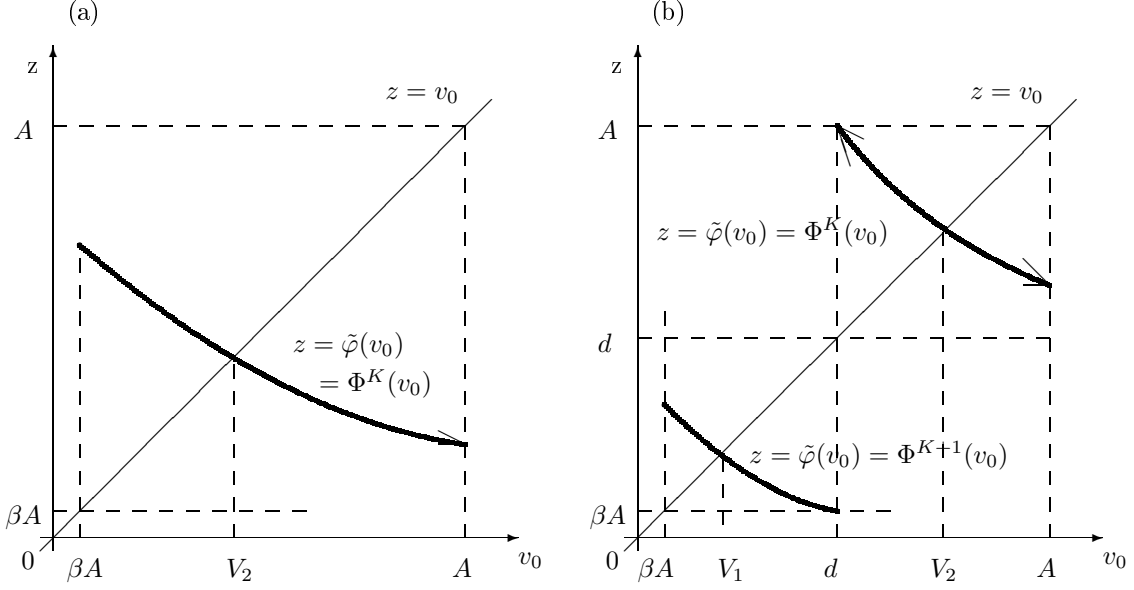 *(α)* $\forall v_0 \in [\beta A, d]$, $\tilde{\varphi}(v_0) = \Phi^{K+1}(v_0) \in [\beta A, d]$;
 (β) $\forall v_0 \in (d, A)$, $\tilde{\varphi}(v_0) = \Phi^K(v_0) \in (d, A)$.
See Fig.14. (Note that, according to the definition of K, $A \in \left(\frac{\beta^K}{1 - \beta^K}, \frac{\beta^{K-1}}{1 - \beta^{K-1}} \right]$; *according to Lemma 2,* $A_{K+1}^* \in \left(\frac{\beta^K}{1 - \beta^K}, \frac{\beta^{K-1}}{1 - \beta^{K-1}} \right].$ *)*

<u>Proof.</u> (a) According to the definition, $d = \frac{A}{\beta^K} - s^* - 1$, where s^* solves (38) under $v_0 = d$. If $d = \beta A$ then

$$\begin{cases} (1 + A - \beta A)(1 - e^{-s^*}) &= s^*; \\ \frac{A}{\beta^K} - s^* - 1 &= \beta A, \end{cases}$$

Figure 14: Graphs of $\tilde{\varphi}(v_0)$.

or equivalently

$$\begin{cases} A & = & \frac{\beta^K(s^*+1)}{1-\beta^{K+1}}; \\ (1 + A - \frac{\beta^{K+1}(s^*+1)}{1-\beta^{K+1}})(1 - e^{-s^*}) & = & s^*. \end{cases}$$

To put it differently, we have $A = A^*_{K+1}$ if $d = \beta A$.

It remains to prove that $d - \beta A = \frac{A}{\beta^K} - s^* - 1 - \beta A$ is a decreasing function of A. Since s^* satisfies equation

$$\left(1 + A - \frac{A}{\beta^K} + s^* + 1\right)(1 - e^{-s^*}) - s^* = 0,$$

$$\frac{ds^*}{dA} = \frac{(1 - \beta^K)(1 - e^{-s^*})^2}{\beta^K e^{-s^*}(e^{-s^*} + s^* - 1)}$$

and

$$\frac{d(d - \beta A)}{dA} = \frac{1}{\beta^K} - \frac{ds^*}{dA} - \beta = \frac{s^* e^{-s^*} + e^{-s^*} - 1 + \beta^K(1 - e^{-s^*})^2 - \beta^{K+1} e^{-s^*}(e^{-s^*} + s^* - 1)}{\beta^K e^{-s^*}(e^{-s^*} + s^* - 1)}.$$

The denominator is obviously positive for $s^* > 0$. The nominator equals zero when $s^* = 0$, its derivative equals

$$e^{-s^*}[-s^* + 2\beta^K(1 - e^{-s^*}) - \beta^{K+1}(2 - 2e^{-s^*} - s^*)].$$

Expression in the square brackets equals zero when $s^* = 0$ and has derivative

$$-1 + 2\beta^K e^{-s^*} - 2\beta^{K+1}e^{-s^*} + \beta^{K+1} \triangleq g(s^*, \beta).$$

Clearly,

$$\frac{\partial g(s^*, \beta)}{\partial s^*} = 2\beta^K e^{-s^*}(\beta - 1) < 0,$$

and finally $g(0, \beta) = -1 + 2\beta^K - \beta^{K+1} < 0$ for all $\beta \in (0, 1)$ because $g(0, 1) = 0$ and $\frac{dg(0,\beta)}{d\beta} = \beta^{K-1}[K(1 - \beta) + K - \beta] > 0$. Therefore $\frac{d(d-\beta A)}{dA} < 0$.

According to Lemma 1, $\tilde{\varphi}$ can coincide with Φ^K or Φ^{K+1} only. In case (a), $\Phi^K(A) < A$ because $\lim_{v_0 \to A} s^* = 0$ (see (31)). Function Φ^K increases as v_0 decreases (Lemma 3), but $\Phi^K(v_0) = A$ when $v_0 = d < \beta A$. Thus, $\forall v_0 \in [\beta A, A)$ $\Phi^K(v_0) < A$, $(K + 1)$ instant jumps are never needed and $\tilde{\varphi} = \Phi^K$.

(b) In this case, $d \geq \beta A$ according to (a). Since $\Phi^K(d) = A$ and Φ^K is a decreasing function (Lemma 3), $\Phi^K(v_0) \geq A$ if $v_0 \in [\beta A, d]$ and $\tilde{\varphi}(v_0) = \Phi^{K+1}(v_0)$, as K jumps are not sufficient. Obviously, $\tilde{\varphi}(d) = \Phi^{K+1}(d) = \beta A$. Now

$$\tilde{\varphi}(\beta A) = \Phi^{K+1}(\beta A) = \Phi^{K+1}(d) - \int_{\beta A}^{d} \frac{d\Phi^{K+1}(v_0)}{dv_0}dv_0 < \Phi^{K+1}(d) + (d - \beta A) = d$$

according to Lemma 3, and statement (α) is proved.

In case (β), $\Phi^K(v_0) < A$, hence $\tilde{\varphi}(v_0) = \Phi^K(v_0)$. We know that $\Phi^K(d) = A$. Using Lemma 3, we conclude that

$$\Phi^K(A) = \Phi^K(d) + \int_{d}^{A} \frac{d\Phi^K(v_0)}{dv_0}dv_0 > A - (A - d) = d.$$

■

Corollary 4 *Theorem 1 and Corollary 1 hold for unclipped cycles.*

Proof. (See Fig.14.) Under conditions (a) of Lemma 4, $\tilde{\varphi}$ has a stable stationary point V_2 coincident with that of Φ^K. (Note that $\Phi^K(A) < A$, so that $V_2 \in [\beta A, A)$.)

Consider case (b) of Lemma 4.

If $v_0 \in [\beta A, d]$ then $\tilde{\varphi} = \Phi^{K+1}$ is a contraction defined on this interval; so that the statement follows.

If $v_0 \in (d, A)$, $\tilde{\varphi}$ has a stable stationary point V_2 coincident with that of Φ^K. (Note that $\Phi^K(A) < A$, hence $d < A = \Phi^K(d)$, so that $V_2 \in (d, A)$.)

Corollary 1 is obvious. ■

Critical cycles.

Remind that a cycle is called critical if $\min_s y(s) = 0$ From (25,28,29) it is clear that the minimum is attained at

$$s_0 = \ln \frac{s_1}{1 - e^{-s_1}}, \tag{39}$$

where s_1 solves (29).

Lemma 5 *Suppose, an unclipped k-cycle exists.*
(a) $y(s_0)$ increases with A.
(b) For cycles of order $k = 1$, $\exists \varepsilon > 0 \, \exists \delta > 0$: $\frac{dy(s_0)}{dA} > \varepsilon$ as soon as $A > \frac{\beta}{1-\beta} + \delta$.
Consequently $y(s_0) \to \infty$ as $A \to \infty$.

Proof. (a) After rewriting (29) in the form

$$\left(1 + A - \frac{\beta^k(s_1 + 1)}{1 - \beta^k} \right)(1 - e^{-s_1}) - s_1 = 0,$$

we obtain:

$$\frac{ds_1}{dA} = \frac{1 - e^{-s_1}}{\frac{\beta^k}{1-\beta^k}(1 - e^{-s_1}) - e^{-s_1}\frac{s_1}{1-e^{-s_1}} + 1} = \frac{(1 - e^{-s_1})^2(1 - \beta^k)}{1 - e^{-s_1} - s_1 e^{-s_1}(1 - \beta^k) - \beta^k e^{-s_1} + \beta^k e^{-2s_1}}. \tag{40}$$

The denominator has derivative (wrt $s_1 > 0$)

$$s_1 e^{-s_1}(1 - \beta^k) + 2\beta^k(e^{-s_1} - e^{-2s_1}) > 0$$

and hence increases starting from 0 when $s_1 = 0$. Therefore $\frac{ds_1}{dA} > 0$.
Since

$$y(s_0) = (1 + A - v_0)e^{-s_0} + s_0 - 1 + v_0 - q = v_0 + s_0 - q \tag{41}$$

we conclude that

$$\frac{dy(s_0)}{dA} = \left(\frac{dv_0}{ds_1} + \frac{ds_0}{ds_1} \right)\frac{ds_1}{dA} = \left(\frac{\beta^k}{1 - \beta^k} + \frac{1 - e^{-s_1} - s_1 e^{-s_1}}{s_1(1 - e^{-s_1})} \right)\frac{ds_1}{dA} > 0.$$

(b) Note that the denominator in (40) is a bounded function of s_1. Thus $\exists \varepsilon > 0 \, \exists \delta_1 > 0$:
$\frac{ds_1}{dA} > \varepsilon$ as soon as $s_1 > \delta_1$, or, equivalently, as soon as $A > \frac{\beta}{1-\beta} + \delta$, where $\delta > 0$ exists
because s_1 monotonically increases with A. Remember that $\lim_{A \to \frac{\beta}{1-\beta}} s_1 = 0$. ■

Lemma 6 *Suppose, all parameters, apart from b, are fixed.*
(a) A critical cycle of order k exists (for some positive value of b) if and only if

$$\frac{\beta^k}{1 - \beta^k} < q \leq q_k^*, \tag{42}$$

where q_k^* is given by (18). The corresponding value of b equals $b_{0,k}$, see (15).
(b) The boundary q_k^* satisfies inequalities

$$\frac{\beta^{k-1}}{1 - \beta^{k-1}} \leq q_k^* < A_k^*. \tag{43}$$

(In case $k = 1$, $q_1^* = +\infty$.)

<u>Proof.</u> (a) Necessity. Let $k > 1$ and suppose a critical cycle of order k exists. Then, if we increase b up to $b^* = A_k^* - q$, this k-cycle (equipped with an asterisk) must remain unclipped (Lemma 5):

$$y^*(s_0^*) = v_0^* + s_0^* - q \geq 0 \tag{44}$$

(see (41)), ie $q \leq v_0^* + s_0^*$. Here $s_0^* = \ln \frac{s_1^*}{1-e^{-s_1^*}}$ (see 39)), s_1^* solves (29) under A_k^* and hence coincides with τ_k defined by (16); v_0^* is defied by (28). Therefore, $v_0^* + s_0^* = q_k^*$.

Obviously, system of equations (28,29,39) and

$$v_0 + s_0 - q = 0$$

(see (41)) must be compatible, ie equation

$$h(s_1) = \frac{\beta^k(s_1 + 1)}{1 - \beta^k} + \ln \frac{s_1}{1 - e^{-s_1}} - q = 0 \tag{45}$$

must have a positive solution. One can easily check that h increases to infinity with s_1, starting from $\lim_{s_1 \to 0} h(s_1) = \frac{\beta^k}{1-\beta^k} - q$. Hence $q > \frac{\beta^k}{1-\beta^k}$.

In case $k = 1$ we put $q_1^* = +\infty$, so that (42) transforms to $q > \frac{\beta}{1-\beta}$, and the proof of the latter inequality remains unchanged.

Before proving sufficiency, we firstly prove part (b).

(b) Let $k > 1$;

$$h \overset{\triangle}{=} q_k^* - \frac{\beta^{k-1}}{1 - \beta^{k-1}} = \frac{\beta^k}{1 - \beta^k}(\tau_k + 1) + \ln \frac{\tau_k}{1 - e^{-\tau_k}} - \frac{\beta^{k-1}}{1 - \beta^{k-1}}$$

$$= \ln \frac{\tau_k}{1 - e^{-\tau_k}} - \alpha(\tau_k + 1) - \alpha(\tau_k + 1)\gamma + \tau_k\gamma,$$

where $\alpha \overset{\triangle}{=} \frac{\beta^{k-1} - \beta^k}{1 - \beta^k}$, $\gamma \overset{\triangle}{=} \frac{\beta^{k-1}}{1 - \beta^{k-1}}$. Using (16), the last expression can be rewritten as

$$h = 1 - \frac{\tau_k}{1 - e^{-\tau_k}} + \ln \frac{\tau_k}{1 - e^{-\tau_k}} + \left(1 - \frac{\tau_k}{1 - e^{-\tau_k}}\right)\gamma + \tau_k\gamma.$$

For $k > 1$ one can easily check that $\gamma \geq \frac{\alpha}{1-2\alpha}$; therefore, since $1 - e^{-\tau_k} - \tau_k e^{-\tau_k} \geq 0$,

$$h \geq 1 - \frac{\tau_k}{1 - e^{-\tau_k}} + \ln \frac{\tau_k}{1 - e^{-\tau_k}} + \frac{1 - e^{-\tau_k} - \tau_k e^{-\tau_k}}{1 - e^{-\tau_k}} \cdot \frac{\alpha}{1 - 2\alpha}$$

$$= 1 - \frac{\tau_k}{1 - e^{-\tau_k}} + \ln \frac{\tau_k}{1 - e^{-\tau_k}} + \frac{1 - e^{-\tau_k} - \tau_k e^{-\tau_k}}{1 - e^{-\tau_k}} \cdot \frac{\tau_k - 1 + e^{-\tau_k}}{3 - 3e^{-\tau_k} - \tau_k - \tau_k e^{-\tau_k}}. \tag{46}$$

(We have used (16) to express α in terms of τ_k.)

During the proof of Lemma 2(c), we established that τ_k increases with $\alpha \in (0, 1/2)$, starting from 0 when $\alpha = 0$. Hence $\tau_k \in (0, \tau)$, where τ is the single positive solution to equation

$$(1 - e^{-\tau})(1 + \frac{1}{2}(\tau + 1)) = \tau.$$

(The solvability was established in the Proof of Lemma 2(c).)

Now the righthand side of (46) is non-negative if $\tau_k \in (0, \tau)$. This statement was accurately checked numerically; the analytical proof is problematic.

The second inequality, to be verified, is obvious:

$$q_k^* - A_k^* = \frac{\beta^k}{1 - \beta^k}(\tau_k + 1) + \ln \frac{\tau_k}{1 - e^{-\tau_k}} - \frac{\beta^{k-1}(\tau_k + 1)}{1 - \beta^k} = \ln[1 + \alpha(\tau_k + 1)] - \alpha(\tau_k + 1) < 0.$$

(a) Sufficiency. Suppose inequalities (42) hold. Then for $b \in [0, A_k^* - q]$ (unclipped) k-cycles exist according to Theorem 4, see Fig.13. (Remember that $A_1^* = q_1^* = +\infty$.) Note that, in case $k > 1$, $q < A_k^*$ due to (b). In this case, for $b = b^* = A_k^* - q$,

$$y^*(s_0^*) = v_0^* + s_0^* - q = q_k^* - q \geq 0$$

(see (44)) and this particular cycle is really unclipped. In case $k = 1$, according to Lemma 5(b), $y(s_0) > 0$ for sufficiently large b. Now, if b decreases then the minimal value of y over a cycle decreases (Lemma 5(a)) and, being continuous, becomes zero, since $y(s_0) < 0$ for the unclipped k-cycle corrresponding to $b = 0$.

To calculate the critical value of b, note that equation (45) has a single positive solution s_1. Now, if we take

$$b = \frac{s_1}{1 - e^{-s_1}} + \frac{\beta^k(s_1 + 1)}{1 - \beta^k} - 1 - q = \frac{s_1}{1 - e^{-s_1}} - \ln \frac{s_1}{1 - e^{-s_1}} - 1$$

then, according to (28,29), the corrresponding cycle will be critical. (One can easily see that $b > 0$.) It remains to notice that equation (45) is identical with (14). ∎

Corollary 5 *Let N be defined by (11). Then critical cycles of orders $k < N$ cannot exist.*

<u>Proof.</u> According to (11), $q \leq \frac{\beta^k}{1 - \beta^k}$, if $n < N$. The statement follows from Lemma 6(a). ∎

Clipped cycles.
<u>Proof of Theorem 1.</u> Let S be the single positive solution to equation

$$(1 + b + S)e^{-S} = 1.$$

Then a continuous trajectory (25) starting from $(y_0 = b, \ v_0 = q - S)$ touches the axis $y = 0$ at a single point, at time moment S.

(a) In case $S > q - \beta A$ it is obvious that starting from any point $(y_0 = b, v_0 \in [\beta A, A))$, the trajectory never touches the axis $y = 0$. The statements follow now from Corollary 4: the mappings φ and $\tilde{\varphi}$ coincide.

(b) Suppose that $S \leq q - \beta A$ and $q - S < V$, where V $(= V_1$ or $V_2)$ is the minimal stationary point of the mapping $\tilde{\varphi}$ (see Lemma 4 and Fig.14). Then, starting from any point $(y_0 = b, v_0 \in [\beta A, A))$, at most $\varphi(\varphi(v_0))$ is such that the further trajectory never touches the axis $y = 0$: see Lemmas 3 and 4. To put it differently, $\varphi^n(v_0) > q - S$ for $n \geq 2$. The required statements again follow from Corollary 4. The mappings φ and $\tilde{\varphi}$ coincide on the domain $[q - S, A)$.

(c) Suppose that $S \leq q - \beta A$, $q - S \geq V_2$, where V_2 is the maximal stationary point of the mapping $\tilde{\varphi}$ (see Lemma 4 and Fig.14). Then, starting from any point $(y_0 = b, v_0 \in [\beta A, A))$ $\forall n \geq 2$, $\varphi^n(v_0) = \varphi(\varphi(v_0))$ because $\varphi(v_0)$, $\varphi(\varphi(v_0)) \leq V_2 \leq q - S$. Note that in terms of Theorem 1, $d < \beta(q + b)$, $V_2 = \varphi(\varphi(v_0))$ is different from (smaller than) V_2 shown on Fig.14.

(d) Suppose that $S \leq q - \beta A$, case (b) (Lemma 4) takes place and $V_1 \leq q - S \leq d$ (see Fig.14). Then, if $v_0 \in [\beta A, d]$, the situation is similar to (c): $\forall n \geq 2$ $\varphi^n(v_0) = \varphi(\varphi(v_0))$, because $\varphi(v_0)$, $\varphi(\varphi(v_0)) \leq V_1 \leq q - S \leq d$.

If $v_0 \in (d, A)$ then the trajectory never touches axis $y = 0$ because $\forall n$ $\varphi^n(v_0) = \tilde{\varphi}^n(v_0) > d \geq q - S$. The statements follow from Corollary 4.

(e) Suppose that $S \leq q - \beta A$, case (b) (Lemma 4) takes place and $d < q - S < V_2$ (see Fig.14). Then situation is similar to (b). Starting from any point $(y_0 = b, v_0 \in [\beta A, A))$, at most $\varphi(\varphi(v_0))$ is such that the further trajectory never touches the axis $y = 0$, the mappings φ and $\tilde{\varphi}$ coincide on the domain $[q - S, A)$ and the required statements follow from Corollary 4. ∎

Corollary 1 is now obvious.

Corollary 6 *If a clipped k-cycle exists then an unclipped k-cycle exists, too. (See (24).)*

<u>Proof.</u> As is clear from the proof of Theorem 1, $0 < S \leq q - \beta A$ and $\varphi(q - S) = \Phi^k(q - S)$. To put it differently, the domain of Φ^k is non-empty, so that the corresponding stationary point V_1 or V_2 (Fig.14) does exist and defines the unclipped k-cycle. ∎

Corollary 7 *The order of a clipped cycle can be N or $N + 1$ only (see (11)).*

<u>Proof.</u> Suppose all parameters are fixed, apart from b. For very small values of b, obviously, only a clipped N-cycle is realised. Conditions when a clipped $(N + 1)$-cycle exists, are left till the next subsecion.

Suppose $N > 1$. When we increase b, k-cycles with $k < N$ appear: see Fig.13. If b is close to $\frac{\beta^k}{1 - \beta^k} - q$ then the k-cycle has a very short continuous part. From the proof of Lemma 5, we have

$$\lim_{b \to \frac{\beta^k}{1 - \beta^k} - q} s_0 = 0 \text{ and } \lim_{b \to \frac{\beta^k}{1 - \beta^k} - q} y(s_0) = \frac{\beta^k}{1 - \beta^k}.$$

(See (39,41). Therefore, using Lemma 5(a) we conclude that all k-cycles remain unclipped indeed. See also Corollary 5. ∎

Effects of the router buffer b.

The goal of this subsection is to justify all the statements of Section 4.

Case $A^*_{N+1} < q$ is trivial: see Fig.13, Lemmas 5,6, Corollary 7 and its proof.

Case $q \leq q^*_{N+1}$. According to Lemma 6, here the $(N+1)$-cycle appears and becomes critical before it extincts at $b = A^*_{N+1} - q$.

Consider the continuous trajectory (25) staring from $(y_0 = 0, v_0 = q)$:

$$\begin{cases} y(r) = e^{-r} + r - 1; \\ v(r) = q + r. \end{cases}$$

Clearly, there is $1 - 1$ correspondance between parameters r and b given by equation

$$e^{-r} + r - 1 = b. \tag{47}$$

The $(N+1)$-cycle cannot be realised if

$$\beta^N(q + r + 1) < y(r) + q = e^{-r} + r - 1 + q = b + q.$$

Let us study the difference

$$\Delta(r) \overset{\triangle}{=} e^{-r} + r - 1 + q - \beta^N(q + r + 1). \tag{48}$$

Since $\frac{d\Delta(r)}{dr} = 1 - e^{-r} - \beta^N$, this difference has a minimum at $r = -\ln(1 - \beta^N)$ (corresponding to $b = C$, see (13)) which equals

$$q(1 - \beta^N) - 2\beta^N - (1 - \beta^N)\ln(1 - \beta^N) = (1 - \beta^N)(q - D),$$

see (12). Since the critical $(N+1)$-cycle exists, we are sure that $q \leq D$ and the values $\underline{b}$ and $\bar{b}$ (20) are well defined. These equal the minimal and the maximal values providing $\Delta(r(b)) = 0$. Here and below, $r(b)$ is the positive solution to (47). Note that the clipped $(N+1)$-cycle appears when $b = \underline{b}$ and becomes critical at $b = b_{0,N+1}$. The value $\bar{b}$ does not play any role because $\bar{b} \geq b_{0,N+1}$.

Case $q^*_{N+1} < q \leq A^*_{N+1}$. Here the $(N+1)$-cycle cannot be critical (Lemma 6). According to Lemma 5, it also cannot be unclipped because unclipped cycle becomes critical when b decreases. Sometimes $(N+1)$-cycles are not realised at all. Firstly, the latter happens if $D < q$. But even if $D \geq q$, it can happen that $\underline{b} > A^*_{N+1} - q$, so that the $(N+1)$-cycle does not exist in view of Corollary 6.

Lemma 7 *Suppose $q^*_{N+1} < q \leq A^*_{N+1}$.*

(a) For a given value of b, the clipped $(N+1)$-cycle exists iff $\Delta(r(b)) \leq 0$ and $b \leq A^*_{N+1} - q$.

(b) $\Delta(r(A^*_{N+1} - q)) > 0$.

(c) Suppose that $D \geq q$. Then $\underline{b} > A^*_{N+1} - q$ iff $C > A^*_{N+1} - q$; $\bar{b} < A^*_{N+1} - q$ iff $C < A^*_{N+1} - q$.

<u>Proof.</u> (a) The necessity is obvious: see Corollary 6 and Fig.13.

Suppose $\Delta(r(b)) \leq 0$ and $b \leq A^*_{N+1} - q$. For the unclipped $(N+1)$-cycle, the minimal value of y is negative; let us denote the corresponding minimal value of v by $\hat{v}$. Then, starting from $(y_0 = 0, v_0 = q)$, the trajectory (25) reaches the level $y = b$, and, after $(N+1)$ instant reductions of v, reaches point $(y = b, v < \hat{v})$. After that, the trajectory goes down up to the axis $y = 0$, and the clipped $(N+1)$-cycle is well defined.

(b) Value $b = A^*_{N+1} - q$ is the largest buffer size when the unclipped $(N+1)$-cycle exists: see Fig.13. The corresponding minimal value y_{min} is negative and, starting from $(y_0 = y_{min}, v_0 = y_{min} + q)$ trajectory (25) reaches level $y = b$ at such value of v that $\beta^N v = b + q$. Therefore, starting from $(y_0 = 0, v_0 = q)$, trajectory (25) reaches level $y = b$ at a smaller value of v, and smaller than $(N+1)$ reductions of v are needed, meaning that $\Delta(r(b)) > 0$.

(c) Obviously, $\underline{b} \leq C \leq \bar{b}$. Thus the necessity is trivial. The sufficiency follows from (b) because $A^*_{N+1} - q \notin [\underline{b}, \bar{b}]$. ∎

Corollary 8 *In case $q^*_{N+1} < q \leq A^*_{N+1}$, $q \leq D$, the value of C cannot equal $A^*_{N+1} - q$.*

The proof follows directly from statement (b), Lemma 7.

Corollary 9 *Suppose N is fixed.*
(a) For all $q \in (q^*_{N+1}, A^*_{N+1}]$ the value of N remains unchanged.
(b) If $D \geq A^*_{N+1}$ then $\forall q \in (q^*_{N+1}, A^*_{N+1}]$ $C > A^*_{N+1} - q$.
(c) If $q^*_{N+1} < D < A^*_{N+1}$ then either $C > A^*_{N+1} - D$ and $\forall q \in (q^*_{N+1}, D]$ $C > A^*_{N+1} - q$, or $C < A^*_{N+1} - D$ and $\forall q \in (q^*_{N+1}, D]$ $C < A^*_{N+1} - q$.
(d) If $\beta \in (0, 1)$ varies, equality $C = A^*_{N+1} - D$ can hold only in the area where $D \leq q^*_{N+1}$.

<u>Proof.</u> (a) The assertion follows from inequalities

$$\frac{\beta^N}{1 - \beta^N} \leq q^*_{N+1} < A^*_{N+1} \leq \frac{\beta^{N-1}}{1 - \beta^{N-1}},$$

see Lemma 2(d) and Lemma 6(b). As usual, $\frac{\beta^{N-1}}{1-\beta^{N-1}} = +\infty$ if $N = 1$.

(b) Clearly, if $q = A^*_{N+1} = \min\{A^*_{N+1}, D\}$ then $C > 0 = A^*_{N+1} - q$. If we decrease q up to q^*_{N+1}, the values of C and A^*_{N+1} remain unchanged and situation $C = A^*_{N+1} - q$ is excluded due to Corollary 8.

(c) The proof is similar to (b): take $q = D = \min\{A^*_{N+1}, D\}$ and reduce its value.

(d) In case $C = A^*_{N+1} - D$ and $D > q^*_{N+1}$ we have a contradiction to (c). ∎

One can show that different situations studied in Lemma 7 and Corollaries 8 and 9 can really take place.

Theorem 2 follows directly from Section 4.

<u>Proof of Proposition 1.</u> According to definition (17), $\delta = \frac{\beta(1+2\beta-\tau)}{1-\beta^2}$, where τ solves equation

$$\frac{\tau(1 + \beta)}{1 + 2\beta + \beta\tau} = 1 - e^{-\tau}.$$

The both functions on the left and on the right increase from zero, and τ is smaller than θ which solves equation $\frac{\theta(1+\beta)}{1+2\beta+\beta\theta} = 1$, ie $\tau < \theta = 2\beta + 1$. Now

$$\frac{\tau(1+\beta)}{1+2\beta+\beta\tau} < 1 - e^{-(2\beta+1)} \implies \tau < \frac{(1+2\beta)(1 - e^{-(2\beta+1)})}{1 + \beta e^{-(2\beta+1)}}$$

and

$$\delta > \frac{\beta}{1-\beta^2} \cdot \frac{(1+2\beta)(\beta+1)e^{-(2\beta+1)}}{1 + \beta e^{-(2\beta+1)}} \to \infty \text{ as } \beta \to 1.$$

■

<u>Proof of Theorem 3.</u>

First we consider the case $b \in [0, b_{0,1}]$. In this case, the cycle is clipped or critical (see Figure 7). According to Condition (b) of Theorem 2, if $q > A_2^*$ the cycle does not have multiple jumps for any size of the buffer. Without loss of generality, we assume that the zero time moment corresponds to the time moment just after the jump (Point A). Recall that we denote the transformed time by s and the original time by t. We denote by S_A the transformed time when the system reaches point A, by S_B the transformed time when the system reaches point B, and so on. Without loss of generality, we assume that $S_A = 0$. We also use the notation: $S_{AB} = S_B - S_A = S_B$, $S_{BC} = S_C - S_B$, and so on.

From (25) we have

$$y(S_C + u) = y_{CD}(u) = e^{-u} + (u - 1), \quad \text{for} \quad u \in [0, S_{CD}],$$

so that

$$y(S_D) = e^{-S_{CD}} + S_{CD} - 1 = b.$$

We note that $v(S_C) = q$. Consequently, $v(S_D) = q + S_{CD}$, $v(S_E) = q + S_{CD} + 1$ and $v(S_A) = \beta(q + S_{CD} + 1)$. Again, from (25) we have

$$y(s) = (1 + q + y(S_A) - v(S_A))e^{-s} + s - 1 + v(S_A) - q,$$

and

$$y(S_B) = [y(S_A) + 1 + q - v(S_A)]e^{-S_{AB}} + [S_{AB} - 1] + v(S_A) - q = 0.$$

Thus, we have the following equation for S_{AB}

$$[b - \beta S_{CD} + (1 - \beta)(1 + q)]e^{-S_{AB}} + S_{AB}$$

$$+\beta S_{CD} - (1 - \beta)(1 + q) = 0.$$

Now, we can calculate the cycle duration in the original and transformed times. Denote these quantities by T_{cycle} and S_{cycle}, respectively. Note that $S_{cycle} = s_1 + 1$ (see (29) with $k = 1$). From equation $v(S_E) = v(S_A) + S_{cycle}$ we obtain

$$S_{cycle} = (1 - \beta)(q + S_{CD} + 1),$$

and, consequently,

$$T_{cycle} = \int_0^{T_{cycle}} dt = \int_0^{S_{cycle}} \left(T + \frac{x(s)}{\mu} \right) ds = T S_{cycle} + \frac{m}{\mu} \left(\frac{B}{m} + \int_{S_A}^{S_B} y(s)ds + \int_{S_C}^{S_D} y(s)ds \right).$$

Next, we calculate the average queue size

$$\bar{x} = \frac{1}{T_{cycle}} \int_0^{T_{cycle}} x(t)dt = \frac{1}{T_{cycle}} \int_0^{S_{cycle}} x(s) \left(T + \frac{x(s)}{\mu} \right) ds$$

$$= \frac{1}{T_{cycle}} \left[mT \left(\int_{S_A}^{S_B} y(s)ds + \int_{S_C}^{S_D} y(s)ds \right) + \frac{m^2}{\mu} \left(\int_{S_A}^{S_B} y^2(s)ds + \int_{S_C}^{S_D} y^2(s)ds \right) + B \left(T + \frac{B}{\mu} \right) \right].$$

Now we calculate the average sending rate

$$\bar{\lambda} = \frac{1}{T_{cycle}} \int_0^{T_{cycle}} \lambda(t)dt$$

Using (2), we have

$$\bar{\lambda} = \frac{1}{T_{cycle}} \int_0^{T_{cycle}} \frac{w(t)}{T + x(t)/\mu} dt = \frac{m}{T_{cycle}} \int_0^{S_{cycle}} v(s)ds$$

$$= \frac{m}{T_{cycle}} \int_0^{S_{cycle}} \left(\beta(q + 1 + S_{CD}) + s \right) ds = \frac{m}{T_{cycle}} \frac{1}{2} (1 - \beta^2)(q + 1 + S_{CD})^2.$$

For the calculation of the average goodput we use the following formula:

$$\bar{g} = \frac{1}{T_{cycle}} \left[\int_{T_A}^{T_D} \lambda(t)dt + \mu \left(T + \frac{B}{\mu} \right) \right] = \frac{m}{T_{cycle}} \left[\int_{S_A}^{S_D} v(s)ds + q + b \right].$$

In case $b \in (b_{0,1}, \infty)$ the cycle is unclipped. Consequently, the calculations of the average quantities are more straightforward than in the previous case and are based on the knowledge of only one parameter S_{cycle}. ∎

Proof of Proposition 2.

If $B \to \infty$ (equivalently, $b \to \infty$), then $s_1 \to \infty$ (see equation (29)). According to Theorem 3, we have

$$T_{cycle} = \frac{m}{\mu} \left[q(s_1 + 1) + \int_0^{s_1} y(s)ds + b \right]$$

$$= \frac{m}{\mu} \left[1 + 2b + 2q - s_1 - (1 + b + q - \frac{\beta(s_1 + 1)}{1 - \beta})e^{-s_1} + \frac{s_1^2}{2} + \frac{\beta(s_1^2 - 1)}{1 - \beta} \right],$$

and, consequently,

$$\Delta = \mu \frac{\frac{1+\beta}{2(1-\beta)}(2s_1 + 1) - 1 - 2b - 2q + s_1 + (1 + b + q - \frac{\beta(s_1+1)}{1-\beta})e^{-s_1} + \frac{\beta}{1-\beta}}{1 + 2b + 2q - s_1 - (1 + b + q - \frac{\beta(s_1+1)}{1-\beta})e^{-s_1} - \frac{\beta}{1-\beta} + \frac{1+\beta}{2(1-\beta)}s_1^2}$$

$$\sim \frac{(2 + \frac{2(1-\beta)}{1+\beta})s_1}{s_1^2} = \frac{4}{1 + \beta} \frac{1}{s_1} \to 0+, \quad \text{as} \quad s_1 \to \infty.$$

■

<u>Proof of Proposition 3.</u> (a) Suppose N is fixed and $q = \frac{\mu T}{m}$ changes, i.e., increases starting from $\frac{\beta^N}{1-\beta^N}$. Using (14),(15) and omitting for brevity N as the power and the index, we obtain:

$$\frac{dB_0}{dm} = m\frac{db_0}{d\theta} \cdot \frac{d\theta}{dq} \cdot \frac{dq}{dm} + b_0$$

$$= m \left[\frac{1 - e^{-\theta} - \theta e^{-\theta}}{(1 - e^{-\theta})^2} \times \frac{\theta - 1 + e^{-\theta}}{\theta} \right] \left[\frac{1}{\frac{1 - e^{-\theta} - \theta e^{-\theta}}{\theta(1 - e^{-\theta})} + \frac{\beta}{1 - \beta}} \right] \left[-\frac{\mu T}{m^2} \right]$$

$$+ \frac{\theta}{1 - e^{-\theta}} - \ln\frac{\theta}{1 - e^{-\theta}} - 1.$$

We used the implicit differentiation theorem for $\frac{d\theta}{dq}$. Note that $\frac{\mu T}{m} = q$ and express q using (14):

$$\frac{dB_0}{dm} = \left[\frac{\theta}{1 - e^{-\theta}} - \ln\frac{\theta}{1 - e^{-\theta}} - 1 \right]$$

$$- \left[\frac{(1 - e^{-\theta} - \theta e^{-\theta})(\theta - 1 + e^{-\theta})}{\theta(1 - e^{-\theta})^2} \times \frac{\ln\frac{\theta}{1 - e^{-\theta}} + \frac{\beta(\theta+1)}{1 - \beta}}{\frac{1 - e^{-\theta} - \theta e^{-\theta}}{\theta(1 - e^{-\theta})} + \frac{\beta}{1 - \beta}} \right].$$

The second square bracket, $f(\frac{\beta}{1-\beta})$, is a monotonous function of $\frac{\beta}{1-\beta}$.

(α) If $(\theta + 1)\frac{1 - e^{-\theta} - \theta e^{-\theta}}{\theta(1 - e^{-\theta})} - \ln\frac{\theta}{1 - e^{-\theta}} \geq 0$ then $f(\cdot)$ does not decrease and hence

$$\frac{dB_0}{dm} \leq \frac{\theta}{1 - e^{-\theta}} - \ln\frac{\theta}{1 - e^{-\theta}} - 1 - f(0)$$

$$= \frac{\theta}{1 - e^{-\theta}} - \ln\frac{\theta}{1 - e^{-\theta}} - 1 - \frac{\ln\frac{\theta}{1-e^{-\theta}} \cdot (\theta - 1 + e^{-\theta})}{(1 - e^{-\theta})}$$

$$= \frac{\theta}{1 - e^{-\theta}} - 1 - \frac{\theta}{1 - e^{-\theta}}\ln\frac{\theta}{1 - e^{-\theta}} = \gamma - 1 - \gamma\ln\gamma < 0,$$

because $\gamma \triangleq \frac{\theta}{1-e^{-\theta}} \in (1, \infty)$ for $\theta > 0$ and function $\gamma - 1 - \gamma\ln\gamma$ has the maximum which is equal to zero at $\gamma = 1$.

(β) If

$$(\theta + 1)\frac{1 - e^{-\theta} - \theta e^{-\theta}}{\theta(1 - e^{-\theta})} - \ln\frac{\theta}{1 - e^{-\theta}} < 0 \tag{49}$$

then $f(\cdot)$ decreases and hence

$$\frac{dB_0}{dm} < \frac{\theta}{1 - e^{-\theta}} - \ln\frac{\theta}{1 - e^{-\theta}} - 1 - \lim_{y\to\infty} f(y)$$

$$= \frac{\theta}{1 - e^{-\theta}} - \ln \frac{\theta}{1 - e^{-\theta}} - 1 - \frac{(1 - e^{-\theta} - \theta e^{-\theta})(\theta - 1 + e^{-\theta})(\theta + 1)}{\theta(1 - e^{-\theta})^2}.$$

Using (49), we have

$$\frac{dB_0}{dm} < \frac{\theta}{1 - e^{-\theta}} - \frac{(\theta + 1)(1 - e^{-\theta} - \theta e^{-\theta})}{\theta(1 - e^{-\theta})} - 1$$

$$- \frac{(1 - e^{-\theta} - \theta e^{-\theta})(\theta - 1 + e^{-\theta})(\theta + 1)}{\theta(1 - e^{-\theta})^2}$$

$$= \frac{3e^{-\theta} + \theta^2 e^{-\theta} + \theta e^{-\theta} - 2 - e^{-2\theta}}{(1 - e^{-\theta})^2} < 0, \text{ if } \theta > 0.$$

Indeed, consider function $g(\theta) = 3e^{-\theta} + \theta^2 e^{-\theta} + \theta e^{-\theta} - 2 - e^{-2\theta}$. Clearly $g(0) = 0$;

$$\frac{dg}{d\theta} = e^{-\theta}[\theta + 2e^{-\theta} - 2 - \theta^2];$$

$$\left.\frac{dg}{d\theta}\right|_{\theta=0} = 0; \qquad \frac{d[\theta + 2e^{-\theta} - 2 - \theta^2]}{d\theta} = 1 - 2e^{-\theta} - 2\theta < 0$$

because the latter function decreases starting from -1 at $\theta = 0$.

Note that

$$\frac{dB_0}{dm} \to 0 - \text{ as } \theta \to 0 + . \tag{50}$$

(b) Obviously, without loss of generality we can put $N = 1$ and prove that $B_{0,N}$ increases as $\beta \in (0,1)$ decreases. Like previously, we omit N as the power and the index. Now again using the implicit differentiation theorem we obtain

$$\frac{dB_0}{d\beta} = m \frac{db_0}{d\theta} \cdot \frac{d\theta}{d\beta}$$

$$= m \left[\frac{(1 - e^{-\theta} - \theta e^{-\theta})(\theta - 1 + e^{-\theta})}{(1 - e^{-\theta})^2 \theta}\right] \left[\frac{-\frac{1+\theta}{(1-\beta)^2}}{\frac{1-e^{-\theta}-\theta e^{-\theta}}{\theta(1-e^{-\theta})} + \frac{\beta}{1-\beta}}\right] < 0.$$

∎

<u>Proof of Corollary 3</u>. The first part follows directly from Proposition 3, if we notice that N remains unchanged on intervals $m \in \left[\frac{\mu T(1-\beta^{N-1})}{\beta^{N-1}}, \frac{\mu T(1-\beta^N)}{\beta^N}\right)$ and increases by 1 at points m_{i+1}.

When $m \to m_N - 0$, q approaches $\frac{\beta^N}{1-\beta^N}$ and θ_N goes to zero (see (15)). According to (15), $b_{0,N} \to 0+$, hence $B_{0,N} = m b_{0,N} \to 0+$. Equality (50) implies that $\frac{dB_{0,N}}{dm} \to 0-$.

Suppose $m \to 0+$, $q \to \infty$, $N = 1$, $\theta_1 \to \infty$. Then $\frac{\ln \frac{\theta_1}{1-e^{-\theta_1}}}{\theta_1} \to 0$ and $\frac{q}{\theta_1} \to \frac{\beta}{1-\beta}$ according to (14). Therefore

$$B_{0,1} = mb_{0,N} = \frac{\mu T}{q}\theta_1 \left(\frac{b_{0,1}}{\theta_1} \right) \to \frac{\mu T(1-\beta)}{\beta}.$$

Consider $B_{0,N}(m_{N-1})$, ie put $q = \frac{\beta^{N-1}}{1-\beta^{N-1}}$ and study equations (14),(15). Let θ_N be the positive solution to

$$\ln \frac{\theta}{1-e^{-\theta}} + \frac{\beta^N}{1-\beta^N} \cdot \theta = \frac{\beta^{N-1}}{1-\beta^{N-1}} - \frac{\beta^N}{1-\beta^N}; \tag{51}$$

then

$$B_{0,N}(m_{N-1}) = \frac{\mu T(1-\beta^{N-1})}{\beta^{N-1}} \left[\frac{\theta_N}{1-e^{-\theta_N}} - \ln \frac{\theta_N}{1-e^{-\theta_N}} - 1 \right].$$

Obviously, $\lim_{N\to\infty} \theta_N = 0$, hence, directly from (51) we obtain:

$$\lim_{N\to\infty} \left[\frac{\ln \frac{\theta_N}{1-e^{-\theta_N}}}{\theta_N} \cdot \frac{\theta_N}{\beta^N} + \frac{\theta_N}{1-\beta^N} \right] = \frac{1}{2} \lim_{N\to\infty} \frac{\theta_N}{\beta^N} = \lim_{N\to\infty} \left[\frac{\beta^{-1}}{1-\beta^{N-1}} - \frac{1}{1-\beta^N} \right] = \frac{1-\beta}{\beta}$$

and finally

$$\lim_{N\to\infty} m_{N-1}B_{0,N}(m_{N-1}) = \lim_{N\to\infty} \left[\left(\frac{\theta_N}{1-e^{-\theta_N}} - \ln \frac{\theta_N}{1-e^{-\theta_N}} - 1 \right) \Big/ \theta_N^2 \right]$$

$$\times \lim_{N\to\infty} \left(\frac{\theta_N}{\beta^N} \right)^2 \beta^2(\mu T)^2 \lim_{N\to\infty} (1-\beta^{N-1})^2 = \frac{1}{8} \left[\frac{2(1-\beta)}{\beta} \right]^2 \beta^2(\mu T)^2 = \frac{1}{2}(1-\beta)^2(\mu T)^2.$$

■

References

[1] M. Allman, V. Paxson and W. Stevens, TCP congestion control, *RFC 2581*, April 1999, available at `http://www.ietf.org/rfc/rfc2581.txt`.

[2] E. Altman, K.E. Avrachenkov and C. Barakat, "A stochastic model of TCP/IP with stationary random losses", in the Proceedings of ACM SIGCOMM 2000, Stockholm, Sweden, also in *Computer Communication Review*, v.30, no.4, pp.231-242, October 2000.

[3] G. Appenzeller, I. Keslassy and N. McKeown, "Sizing Router Buffers", ACM SIGCOMM '04, Portland, Oregon, September 2004. Also in Computer Communication Review, Vol. 34, No. 4, pp. 281-292, October 2004.

[4] K. Avrachenkov, U. Ayesta and A. Piunovskiy, "Optimal choice of the buffer size in the Internet routers", In Proceedings of IEEE CDC/ECC 2005.

[5] K. Avrachenkov, U. Ayesta, E. Altman, P. Nain, and C. Barakat, "The effect of router buffer size on the TCP performance", in the Proceedings of LONIIS workshop, St. Petersburg, Jan. 29 - Feb. 1, 2002.

[6] S. Bohacek, J.P. Hespanha, J. Lee, and K. Obraczka, "A hybrid systems modeling framework for fast and accurate simulation of data communication networks", in Proceedings of ACM SIGMETRICS 2003, pp.58-69.

[7] M. Christiansen, K. Jeffay, D. Ott and F. Donelson Smith, "Tuning RED for Web Traffic", *IEEE ACM Transactions on Networking*, v.9, no.3, pp.249-264, June 2001. An earlier version appeared in Proc. of ACM Sigcomm 2000.

[8] A. Dhamdhere, H. Jiang, and C. Dovrolis, "Buffer Sizing for Congested Internet Links", in the Proceedings of IEEE Infocom, Miami FL, March 2005.

[9] S. Floyd, "HighSpeed TCP for Large Congestion Windows", *RFC 3649*, December 2003, available at `http://www.ietf.org/rfc/rfc3649.txt`.

[10] S. Floyd and V. Jacobson, "Random Early Detection Gateways for Congestion Avoidance", *IEEE/ACM Transactions on Networking*, v.1, no.4, pp.397–413, 1993.

[11] S. Gorinsky, A. Kantawala, and J. Turner, "Link buffer sizing: a new look at the old problem", in Proceedings of IEEE Symposium on Computers and Communications (ISCC 2005), June 2005.

[12] T. Hatano, M. Fukuhara, H. Shigeno, and K. Okada, "TCP-friendly SQRT TCP for high speed networks", in *Proceedings of APSITT 2003*, pp.455-460, November 2003.

[13] J.P. Hespanha, S. Bohacek, K. Obraczka, and J. Lee, "Hybrid modeling of TCP congestion control", In *Hybrid Systems: Computation and Control*, LNCS v.2034, pp.291-304, 2001.

[14] V. Jacobson, Congestion avoidance and control, *ACM SIGCOMM'88*, August 1988.

[15] T. Kelly, "Scalable TCP: Improving performance in highspeed wide area networks", *Computer Comm. Review*, v.33, no.2, pp.83-91, 2003.

[16] M. May, J. Bolot, C. Diot and B. Lyles, "Reasons Not to Deploy RED", in Proceedings of 7th International Workshop on Quality of Service (IWQoS'99), June 1999, London, UK.

[17] R. Morris, "Scalable TCP congestion control", In Proceedings of IEEE INFOCOM 2000, March 2000, Tel-Aviv, Israel.

[18] "Network Simulator, Ver.2, (NS-2) Release 2.18a", Available at: http://www.isi.edu/nsnam/ns/index.html.

[19] A.B. Piunovskiy, *Optimal Control of Random Sequences in Problems with Constraints.* Kluwer Academic Publishers: Dordrecht, 1997.

[20] J. Postel, User Datagram Protocol, *RFC 768*, August 1980, available at http://www.ietf.org/rfc/rfc0768.txt.

[21] K. Ramakrishnan, S. Floyd and D. Black, The Addition of Explicit Congestion Notification (ECN) to IP, *RFC 3168*, September 2001, available at http://www.ietf.org/rfc/rfc3168.txt.

[22] I. Rhee and L. Xu, "CUBIC: A new TCP-friendly High-Speed TCP variant", in *Proceedings of PFLDnet 2005 workshop*, February 2005.

[23] C. Villamizar and C. Song, "High Performance TCP in the ANSNET", *ACM SIGCOMM Computer Communication Review*, v.24, no.5, pp.45–60, November 1994.

[24] G. Vu-Brugier, R.S. Stanojevic, J. Leith, and R.N. Shorten, "A critique of recently proposed buffer-sizing strategies", *ACM SIGCOMM Computer Communication Review*, v.37(1), pp.43-48, 2007.

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Rapport technique